DASSAULT SYSTEMES

3S SOLIDWORKS

THEORETICAL MANUAL
SOLIDWORKS Simulation

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Introduction
Why Finite Elements?

An investigator seeking the solution of the partial differential equations which govern the behavior of deformable bodies soon discovers that few exact analytical descriptions are available and that those that are available are very much limited in applicability. Solutions are generally obtainable only for regions having certain regular geometric shapes (circles, rectangles, spheres, etc.) and then only for restricted boundary conditions[1-3] The need for results for more complex structures leads to the use of approximate methods of solution.

A number of different approximate methods have been devised since the beginning of the twentieth century. One of the earliest [4] replaces the goal of obtaining a continuously varying solution distribution by that of obtaining values at a finite number of discrete grid or nodal points. The differential equations are replaced by finite difference equations, which, together with appropriate boundary conditions expressed in difference form, yield a set of simultaneous linear equations for the nodal values. An alternative approximate method, the Rayleigh-Ritz method [5] introduced almost at the same time, seeks to expand the solution of the differential equations in a linear series of known functions. The coefficients multiplying these functions are obtained by requiring the satisfaction of the equivalent variational formulation of the problem and are, again, the solution of a set of simultaneous linear equations. These methods have extended the range of problems that may be considered but have been found to be limited by the extreme difficulty involved in applying them to even more complex shapes. The need to analyze the complicated swept-wing and delta-wing structures of high speed aircraft was the impetus which led to the development of the finite element method.

It is common in the traditional analysis of complicated building structures to divide them into pieces whose behavior under general states of deformation or loading is more readily available. The pieces are then reattached subject to conditions of equilibrium or compatibility. The slope- deflection method [6] in statically indeterminate rigid-frame analysis is an example of such an approach. Attempts at rational analysis of wing-structures initially took the same physically motivated path with, however, the improvements of matrix formulations and the use of electronic digital computers. Methods based on Castigliano's theorems were devised for the calculation of flexibility matrices for obtaining deflections from forces and stiffness matrices for the determination of forces from displacements. The former matrices were used in "force" methods of analysis while the latter were used in "displacement" methods.

An explosion in the development of the finite element methods occurred in the years subsequent to 1960 when it was realized that the method, whether based on forces or displacements, could be interpreted as an application of the Rayleigh-Ritz method. This was first suggested for two dimensional continua by R. Courant,[8] who proposed the division of a domain into triangular regions with the desired functions continuous over the entire domain replaced by piecewise continuous approximations within the triangles. The use of flexibility matrices was found to imply the implementation of the principle of minimum complementary energy while stiffness matrices imply the principle of minimum potential energy. The use of this approach permits the investigation of such topics as the continuity requirements for the piecewise

approximations and convergence rates obtained with increasing numbers of elements or with increasing complexity of functional representation. It also allows stiffness or flexibility matrices to be calculated from a conceptually simpler mathematical viewpoint, while indicating the possibility of using variational principles in which both forces and displacements are varied to produce "hybrid" elements. Despite the possible advantage of hybrid elements for some problems, solutions based upon the principle of minimum potential energy and displacement approximations have become dominant for the simple reason that the associated computer software is more universally applicable and requires the least interaction between machine and operator.

In recent years the finite element method has been applied to mechanics problems other than those of structural analysis, i.e., fluid flow and thermal analysis. It has been extended to permit the solution of nonlinear as well as linear problems, those of large deformation geometric nonlinearity and/or material property nonlinearity, for example. It is hard to think of any field in which finite elements are not extensively used to provide answers to problems which would have been unsolvable only a few years ago.

Chapter 1. Fundamental Relations for Linearly Elastic Solids

Problems in solid mechanics deal with states of stress, strain and displacement in deformable solids. The basic relationships which govern these states and which are the basis for finite element applications are summarized below. The discussion is limited to states of small displacements and rotations and rotations to linear elastic materials. A more complete exposition may be found in a number of texts. [1, 9]

1.1. Stresses

1.1.1. Stress Matrix

External loading on the surface of a deformable body is assumed to be transmitted into the interior by the pressure of one part of the body on an adjacent portion. If such a body is divided by a plane having a given orientation in space (Fig. l a) and a region about a point P on the cut surface is considered, the pressure forces on this region may be resolved into a resultant moment vector $\Delta\vec{M}$ and a resultant force vector $\Delta\vec{P}$ (Fig. 1 b). As the region considered is decreased in size about the point, these resultant vectors decrease in magnitude and their directions will vary. In the limit it is assumed that the ratio of the force vector and the area upon which it acts, the stress vector, approaches a limit $\vec{t}$ (Fig. lc), while the ratio of the moment vector and the area, the couple stress vector, vanishes, i.e.

$$\lim_{\Delta A \to 0} \frac{\Delta\vec{P}}{\Delta A} = \vec{t}$$

Equation 1-1a

$$\lim_{\Delta A \to 0} \frac{\Delta\vec{M}}{\Delta A} = 0$$

Equation 1-1b

a. Plane dividing Body into Two Parts

b. Resultant Force and Moment Vectors on region of area ΔA about O in plane

c. Limiting stress vector at point O in plane
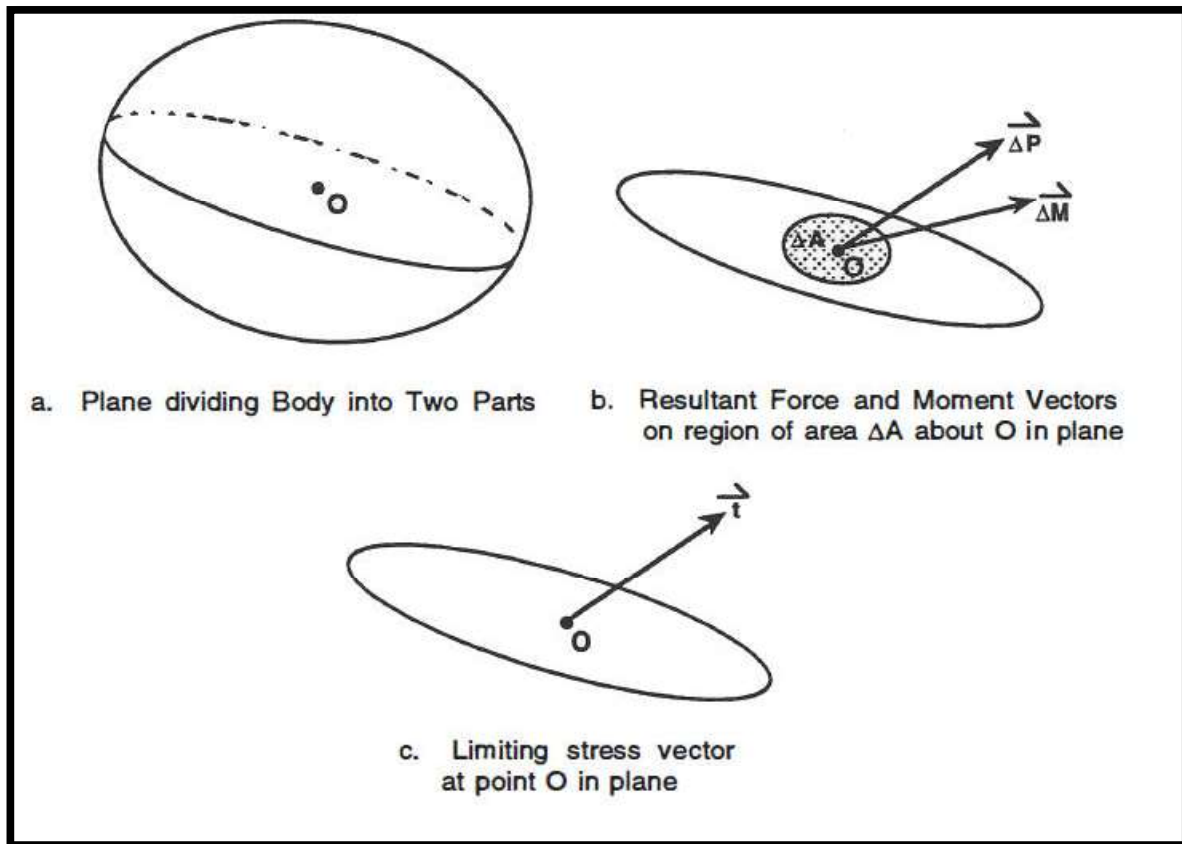
FIGURE 1 THE STRESS VECTOR AT A POINT

The stress vector $\vec{t}$ at a point in the body is a function of the orientation of the plane on which it acts and is related to the components of the stress vectors on three perpendicular planes passing through the point. The set of nine components, called the stress matrix, defines the state of stress at a point. In Cartesian coordinates these nine components are

$$S = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Equation 1-2

The first subscript denotes the direction of the outwardly directed normal to the plane on which the stress component acts while the second subscript denotes the direction of the s tress component. These are shown in Fig. 2 acting on faces for which the outwardly directed normal is in the positive direction of the coordinate axis. On the remaining faces for which the outwardly directed normal is in the opposite direction, the stress component directions are reversed. Conditions of moment equilibrium of forces about a point require symmetry of the stress matrix, i.e.

$$\sigma_{xy} = \sigma_{yx}$$
$$\sigma_{yz} = \sigma_{zy}$$
$$\sigma_{zx} = \sigma_{xz}$$

Equation 1-3
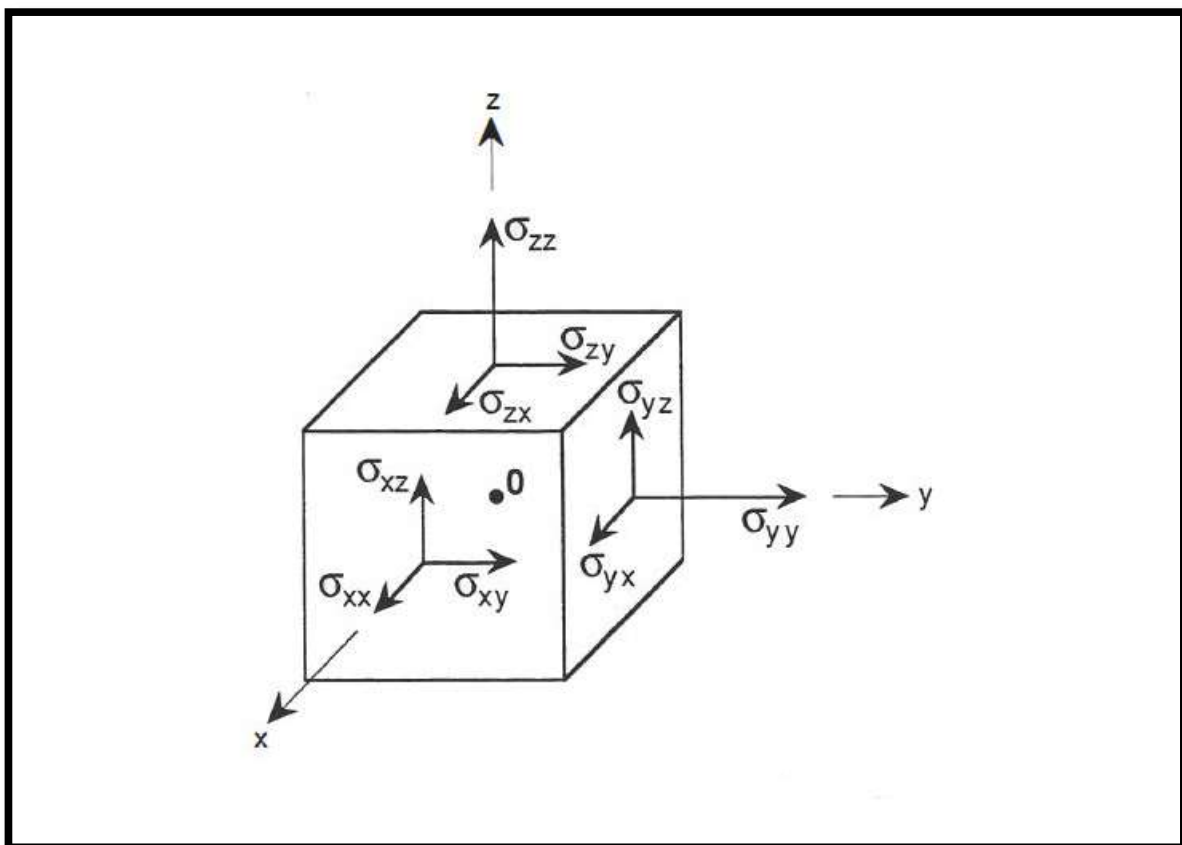
FIGURE 2 STRESS VECTOR COMPONENTS ON THREE PERPENDICULAR PLANES ABOUT POINT 0

If the outwardly directed normal to the plane through point O (Fig. 3) is

$$n = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

Equation 1-4

the stress vector on that plane is given by

$$t_n = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = S^T n = \begin{Bmatrix} \sigma_{xx}n_x + \sigma_{yx}n_y + \sigma_{zx}n_z \\ \sigma_{xy}n_x + \sigma_{yy}n_y + \sigma_{zy}n_z \\ \sigma_{xz}n_x + \sigma_{yz}n_y + \sigma_{zz}n_z \end{Bmatrix}$$

Equation 1-5
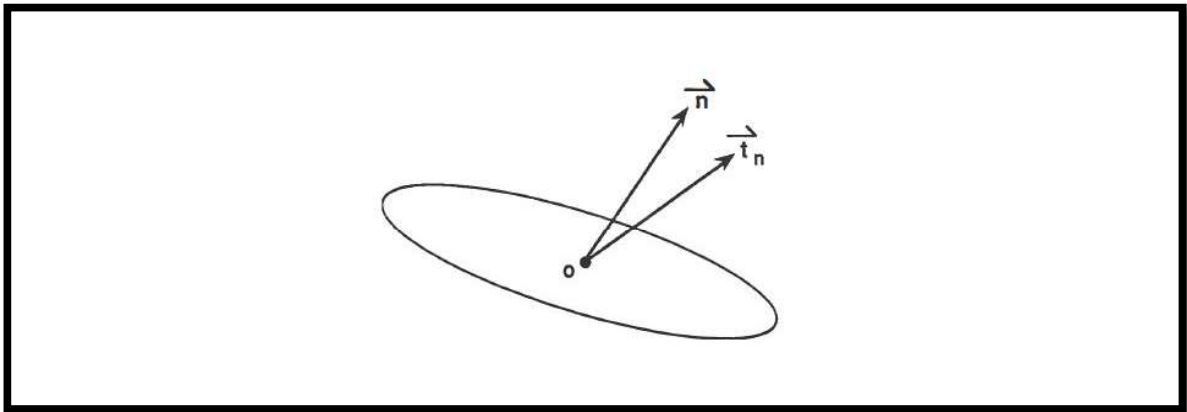
FIGURE 3 STRESS VECTOR IN PLANE WITH NORMAL VECTOR N

1.1.2. Rotated Coordinate Systems

The stress matrix has been defined with respect to a given coordinate system x, y, z. If a second set of Cartesian coordinates x', y', z' having the same origin but different orientation is introduced, the two systems of coordinates are related by (Fig. 4)

$$x' = Nx$$

Equation 1-6

with

$$x = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad x' = \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}$$

Equation 1-7
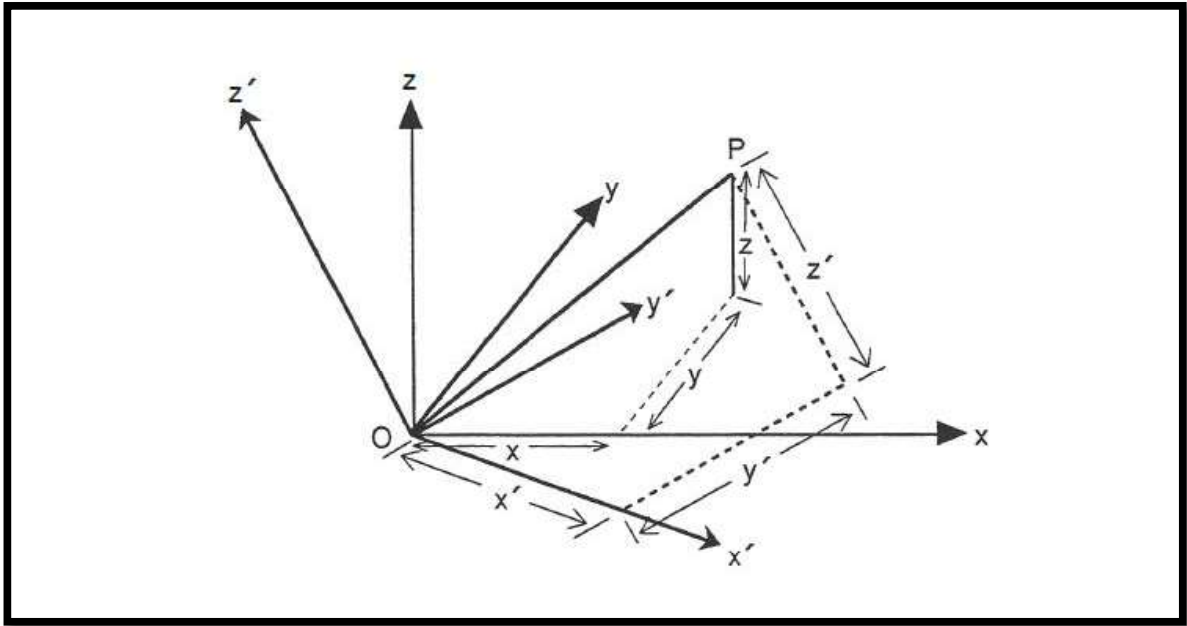
FIGURE 4 COMPONENT OF A VECTOR IN ROTATED CARTESIAN COORDINATE SYSTEMS

and

$$N = \begin{bmatrix} n_{x'x} & n_{x'y} & n_{x'z} \\ n_{y'x} & n_{y'y} & n_{y'z} \\ n_{z'x} & n_{z'y} & n_{z'z} \end{bmatrix}$$

Equation 1-8

where $n_{i'j}$ is the cosine of the angle between the primed i'-axis and the unprimed j-axis. The relationship

$$N^{-1} = N^{T}$$

Equation 1-9

holds for this matrix. The stress matrix with respect to the second set of Cartesian axes is expressed by

$$S' = NSN^{T}$$

Equation 1-10

It is sometimes more convenient to speak of the six independent stress components which comprise the stress matrix

$$\sigma = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

Equation 1-11

The transformation relation under a rotation of the coordinate system given by Eq. (1.6) then becomes

$$\sigma' = T_\sigma \sigma$$

Equation 1-12

with

$$\sigma' = \begin{Bmatrix} \sigma_{x'x} \\ \sigma_{y'y} \\ \sigma_{z'z} \\ \sigma_{x'y} \\ \sigma_{y'z} \\ \sigma_{z'x} \end{Bmatrix}$$

Equation 1-13

and

$$T_\sigma = \begin{bmatrix} n_{x'x}^2 & n_{x'y}^2 & n_{x'z}^2 & 2n_{x'x}n_{x'y} & 2n_{x'y}n_{x'z} & 2n_{x'z}n_{x'x} \\ n_{y'x}^2 & n_{y'y}^2 & n_{y'z}^2 & 2n_{y'x}n_{y'y} & 2n_{y'y}n_{y'z} & 2n_{y'z}n_{y'x} \\ n_{z'x}^2 & n_{z'y}^2 & n_{z'z}^2 & 2n_{z'x}n_{z'y} & 2n_{z'y}n_{z'z} & 2n_{z'z}n_{z'x} \\ n_{x'x}n_{y'x} & n_{x'y}n_{y'y} & n_{x'z}n_{y'z} & n_{x'x}n_{y'y} + n_{x'y}n_{y'x} & n_{x'y}n_{y'z} + n_{x'z}n_{y'y} & n_{x'z}n_{y'x} + n_{x'x}n_{y'z} \\ n_{y'x}n_{z'x} & n_{y'y}n_{z'y} & n_{y'z}n_{z'z} & n_{y'x}n_{z'y} + n_{y'y}n_{z'x} & n_{y'y}n_{z'y} + n_{y'z}n_{z'z} & n_{y'z}n_{z'x} + n_{y'x}n_{z'z} \\ n_{z'x}n_{x'x} & n_{z'y}n_{x'y} & n_{z'z}n_{x'z} & n_{z'x}n_{x'y} + n_{z'y}n_{x'x} & n_{z'y}n_{x'z} + n_{z'z}n_{x'y} & n_{z'z}n_{y'x} + n_{z'x}n_{x'z} \end{bmatrix}$$

Equation 1-14

If the coordinate axes rotate through an angle θ about a coordinate axis, say the z-axis, (Fig. 5) the matrix N becomes

$$N = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equation 1-15
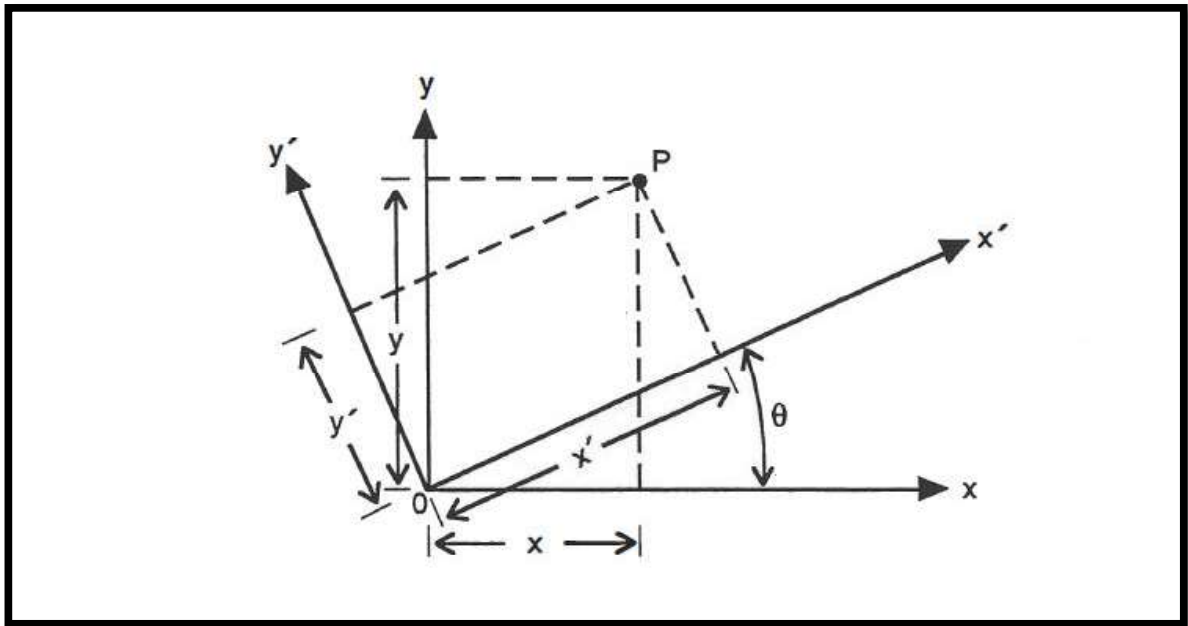
FIGURE 5 COORDINATE SYSTEMS ROTATED ABOUT A COMMON AXIS

and T_σ is given by

$$T_\sigma = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 2\sin\theta\cos\theta & 0 & 0 \\ \sin^2\theta & \cos^2\theta & 0 & -2\sin\theta\cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & 0 & \cos^2(-\sin^2\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Equation 1-16

1.1.3. Principal Stresses

For certain coordinate axis rotations the stress matrix becomes diagonal so that shear stresses vanish. The stress vectors on the three faces perpendicular to the coordinate axes are normal to the surface on which they act (Fig. 6).
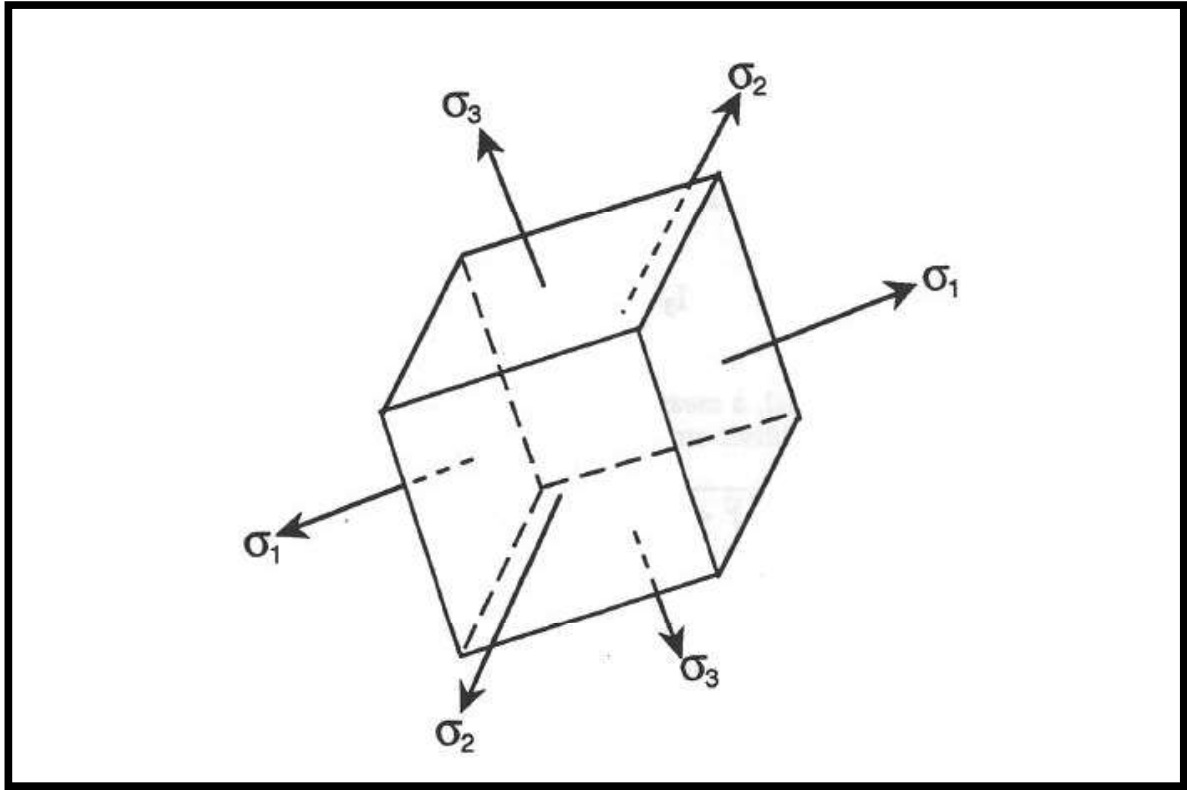
FIGURE 6 PRINCIPAL STRESS COMPONENTS

The three diagonal stress components σ_i are called principal stresses and their corresponding directions are called the principal directions. They are given by the solution of the sets of homogeneous equations

$$[S^T - \sigma_i I]n_i = 0 \qquad i = 1, 2, 3$$

Equation 1-17

where $\mathbf{n_i}$ is the vector defining the direction of the principal stress σ_i. Thus the three values of the principal stresses are the solution of the determinantal equation

$$\det|S^T - \sigma_i I| = 0$$

Equation 1-18

or by the cubic equation

$$\sigma_i^3 - I_1\sigma_i^2 + I_2\sigma_i - I_3 = 0$$

Equation 1-19

where the coefficients are stress invariants independent of the chosen coordinate system and are defied by

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3$$

Equation 1-20 a

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}\sigma_{yx} - \sigma_{yz}\sigma_{zy} - \sigma_{zx}\sigma_{xz} = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

Equation 1-20 b

$$I_3 = \det|S| = \sigma_1\sigma_2\sigma_3$$

Equation 1-20 c

For an isotropic material, a measure of stress intensity required for the material to yield and become plastic is the von Mises stress given by

$$\sigma_M = \sqrt{\frac{1}{2}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)\right]}$$

Equation 1-21 a

This stress is related to the octahedral shearing stress, the shear stress on a plane making equal angles with respect to the principal axes, by

$$\sigma_M = \sqrt{3}\tau_{oct}$$

Equation 21-1 b

1.1.4. Equations of Equilibrium

The six stress components are not arbitrary but must satisfy the force equilibrium equations (Fig. 7)
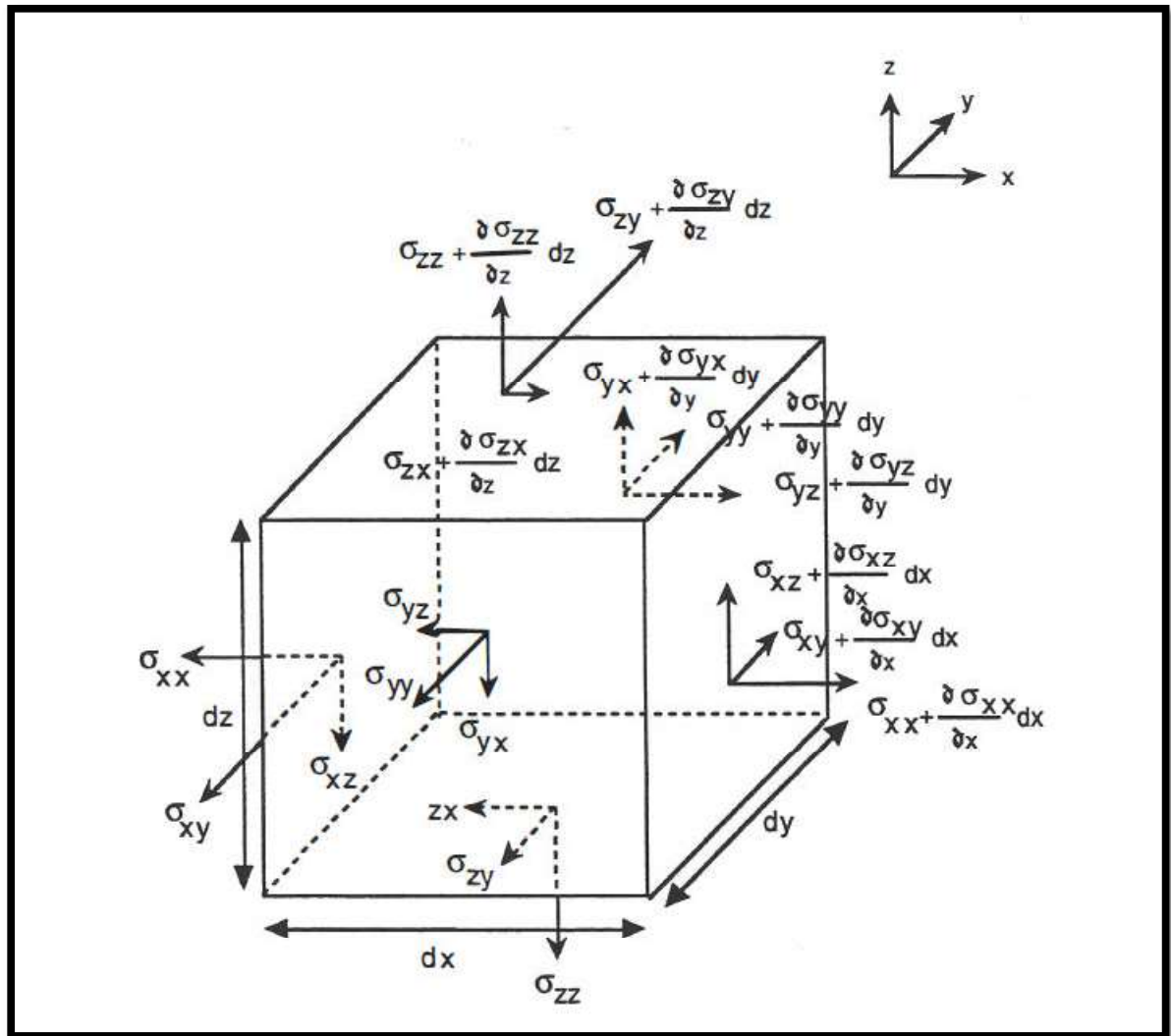
FIGURE 7

$$\frac{\partial \sigma_{xx}}{\partial \sigma_x} + \frac{\partial \sigma_{yx}}{\partial \sigma_y} + \frac{\partial \sigma_{zx}}{\partial \sigma_z} + f_x = 0$$

Equation 1-22 a

$$\frac{\partial \sigma_{xy}}{\partial \sigma_x} + \frac{\partial \sigma_{yy}}{\partial \sigma_y} + \frac{\partial \sigma_{zy}}{\partial \sigma_z} + f_y = 0$$

Equation 1-22 b

$$\frac{\partial \sigma_{xz}}{\partial \sigma_x} + \frac{\partial \sigma_{yz}}{\partial \sigma_y} + \frac{\partial \sigma_{zz}}{\partial \sigma_z} + f_z = 0$$

Equation 1-22 c

Where f_x, f_x, f_x are the components of the body force vector (force per unit volume) $\vec{f}$.

1.2. Strains

1.2.1. Strain Matrix

The deformation state at a point in a deformed body is defined by the strain matrix

$$E = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\varepsilon_{xy} & \frac{1}{2}\varepsilon_{xz} \\ \frac{1}{2}\varepsilon_{yx} & \varepsilon_{yy} & \frac{1}{2}\varepsilon_{yz} \\ \frac{1}{2}\varepsilon_{zx} & \frac{1}{2}\varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

Equation 1-23

in which the diagonal strain components are a measure of the relative change of length of lines originally in the directions of the coordinate axes while the off-diagonal components are symmetric and are a measure of the change of angle between two lines originally in the direction of the coordinate axes (Fig. 8). For small strains and rotations, the relationships between the strain components and the components of the displacement vector

a. Undeformed Element

b. Deformed Element
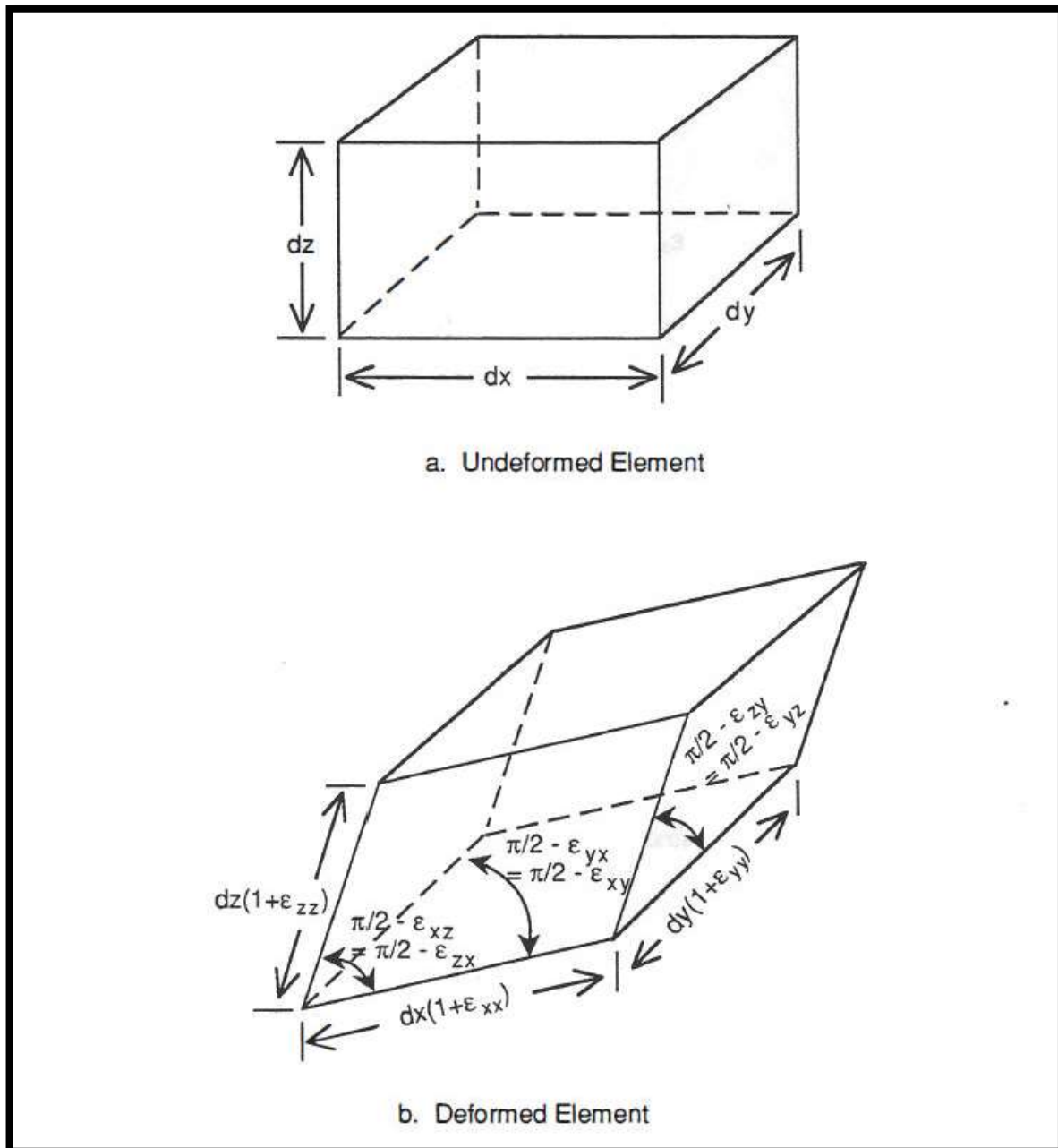
FIGURE 8 INTERPRETATION OF A SMALL STRAIN COMPONENTS

$$u = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$$

Equation 1-24

are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial_x} \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial_y} \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial_z}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\partial u_y}{\partial_x} + \frac{\partial u_x}{\partial_y}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{\partial u_z}{\partial_y} + \frac{\partial u_y}{\partial_z}$$

$$\varepsilon_{zx} = \varepsilon_{xz} = \frac{\partial u_x}{\partial_z} + \frac{\partial u_z}{\partial_x}$$

Equation 1-25

1.2.2. Rotated Coordinate Axes

Under a change of Cartesian coordinate systems at the point, the strain matrix has a transformation similar to that of the stress matrix, i.e.

$$E' = NEN^T$$

Equation 1-26

In terms of the independent strain component matrix

$$\varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}$$

Equation 1-27

the transformation becomes

$$\varepsilon' = T_\varepsilon \varepsilon$$

Equation 1-28

with

$$\varepsilon' = \begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{z'z'} \\ \varepsilon_{x'y'} \\ \varepsilon_{y'z'} \\ \varepsilon_{z'x'} \end{Bmatrix}$$

Equation 1-29

and

$$T_\varepsilon = \begin{bmatrix}
n_{x'x}^2 & n_{x'y}^2 & n_{x'z}^2 & n_{x'x}n_{x'y} & n_{x'y}n_{x'z} & n_{x'z}n_{x'x} \\
n_{y'x}^2 & n_{y'y}^2 & n_{y'z}^2 & n_{y'x}n_{y'y} & n_{y'y}n_{y'z} & n_{y'z}n_{y'x} \\
n_{z'x}^2 & n_{z'y}^2 & n_{z'z}^2 & n_{z'x}n_{z'y} & n_{z'y}n_{z'z} & n_{z'z}n_{z'x} \\
2n_{x'x}n_{y'x} & 2n_{x'y}n_{y'y} & 2n_{x'z}n_{y'z} & n_{x'x}n_{y'y}+n_{x'y}n_{y'x} & n_{x'y}n_{y'z}+n_{x'z}n_{y'y} & n_{x'z}n_{y'x}+n_{x'x}n_{y'z} \\
2n_{y'x}n_{z'x} & 2n_{y'y}n_{z'y} & 2n_{y'z}n_{z'z} & n_{y'x}n_{z'y}+n_{y'y}n_{z'x} & n_{y'y}n_{z'z}+n_{y'z}n_{z'y} & n_{y'z}n_{z'x}+n_{y'x}n_{z'z} \\
2n_{z'x}n_{x'x} & 2n_{z'y}n_{x'y} & 2n_{z'z}n_{x'z} & n_{z'x}n_{x'y}+n_{z'y}n_{x'x} & n_{z'y}n_{x'z}+n_{z'z}n_{x'y} & n_{z'z}n_{y'x}+n_{z'x}n_{x'z}
\end{bmatrix}$$

Equation 1-30

T_ε is related to T_σ by

$$T_\varepsilon = (T_\sigma^T)^{-1}$$

If the rotation is about a coordinate axis, say the z axis,

$$T_\varepsilon = \begin{bmatrix}
\cos^2\theta & \sin^2\theta & 0 & 2\sin\theta\cos\theta & 0 & 0 \\
\sin^2\theta & \cos^2\theta & 0 & -2\sin\theta\cos\theta & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-\sin\theta\cos\theta & \sin\theta\cos\theta & 0 & \cos^2(-\sin^2\theta) & 0 & 0 \\
0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \\
0 & 0 & 0 & 0 & \sin\theta & \cos\theta
\end{bmatrix}$$

Equation 1-31

1.2.3. Principal Strains

As in the case of stress, the strain matrix becomes diagonal for a particular set of axes, the principal strain directions, which are the solution of homogeneous equation

$$[E^T - \varepsilon_i I]n_i = 0 \quad i = 1, 2, 3$$

Equation 1-32

The three principal strains are given by the solution of the cubic equation

$$\varepsilon_i^3 - J_1 \varepsilon_i^2 + J_2 \varepsilon_i - J_3 = 0$$

Equation 1-33

in which the invariants coefficients are defined as

$$J_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

Equation 1-34 a

$$J_2 = \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} - \frac{1}{4}\left(\varepsilon_{xy}\varepsilon_{yx} + \varepsilon_{yz}\varepsilon_{zy} + \varepsilon_{zx}\varepsilon_{xz}\right) = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1$$

Equation 1-34 b

$$J_3 = \det|E| = \varepsilon_1\varepsilon_2\varepsilon_3$$

Equation 1-34 c

1.3. Stress-Strain Relations

1.3.1. Anisotropic Material

The relationship between the components of stress and strain is the generalized Hooke's law given by the linear equation

$$\sigma = C(\varepsilon - \overline{A}\Delta T)$$

Equation 1-35

in which C is the symmetric elastic-coefficient matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{sym} & & & & & C_{66} \end{bmatrix}$$

Equation 1-36

$\overline{A}$ is the thermal expansion coefficient matrix

$$\overline{A} = \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{zz} \\ \alpha_{xy} \\ \alpha_{yz} \\ \alpha_{zx} \end{Bmatrix}$$

Equation 1-37

and ΔT is the difference between the actual temperature and the uniform temperature at which the body is stress free. The relation may also be expressed by the inverted form

$$\varepsilon = F\sigma + A\Delta T$$

Equation 1-38

in which

$$F = C^{-1}$$

Equation 1-39

Under a rigid body of rotation of the coordinate system the matrix C transforms as

$$C' = T_\sigma C T_\sigma^T$$

Equation 1-40a

while F transforms as

$$F = T_\varepsilon F T_\varepsilon^T$$

Equation 1-40b

For an orthotropic material, one with three preferred material axes and with the material axes coinciding with the coordinate axes, the matrices F and $\overline{A}$ are defined by

$$F = \begin{bmatrix} \dfrac{1}{E_{xx}} & -\dfrac{v_{xy}}{E_{yy}} & -\dfrac{v_{xz}}{E_{zz}} & 0 & 0 & 0 \\[2ex] -\dfrac{v_{yx}}{E_{xx}} & \dfrac{1}{E_{yy}} & -\dfrac{v_{yz}}{E_{zz}} & 0 & 0 & 0 \\[2ex] -\dfrac{v_{zx}}{E_{xx}} & -\dfrac{v_{zy}}{E_{yy}} & \dfrac{1}{E_{zz}} & 0 & 0 & 0 \\[2ex] 0 & 0 & 0 & \dfrac{1}{G_{xy}} & 0 & 0 \\[2ex] 0 & 0 & 0 & 0 & \dfrac{1}{G_{yx}} & 0 \\[2ex] 0 & 0 & 0 & 0 & 0 & \dfrac{1}{G_{zx}} \end{bmatrix}$$

Equation 1-41

in which

$$\frac{v_{xy}}{E_{yy}} = \frac{v_{yx}}{E_{xx}} \qquad \frac{v_{xz}}{E_{zz}} = \frac{v_{zx}}{E_{xx}} \qquad \frac{v_{yz}}{E_{zz}} = \frac{v_{zy}}{E_{yy}}$$

Equation 1-42

and

$$\overline{A} = \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{zz} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Equation 1-43

Then the elastic constant matrix C may be written as

$$C = \begin{bmatrix} (1 - v_{yz}v_{zx})S_{xx} & (v_{xy} - v_{xz}v_{zy})S_{xx} & (v_{xz} - v_{xy}v_{yz})S_{xx} & 0 & 0 & 0 \\ (v_{yx} - v_{yz}v_{zx})S_{yy} & (1 - v_{zx}v_{xz})S_{yy} & (v_{yz} - v_{yx}v_{xz})S_{yy} & 0 & 0 & 0 \\ (v_{zx} - v_{zy}v_{yx})S_{zz} & (v_{zy} - v_{zx}v_{xy})S_{zz} & (1 - v_{xy}v_{yx})S_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{yz} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{zx} \end{bmatrix}$$

Equation 1-44

with

$$\frac{S_{xx}}{E_{xx}} = \frac{S_{yy}}{E_{yy}} = \frac{S_{zz}}{E_{zz}} = (1 - v_{xy}v_{zx} - v_{yz}v_{zy} - v_{xy}v_{yx} - v_{xy}v_{yz}v_{zx} - v_{yx}v_{xz}v_{zy})^{-1}$$

Equation 1-45

If only the terms above the principal diagonal of F are defined, the inverted matrix may be written as

$$C = \begin{bmatrix} S_{xx}\left(1 - v_{yx}^2 \frac{E_{xx}}{E_{zz}}\right) & S_{xx}\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) & S_{xx}v_{xz} + v_{xu}v_{yz} & 0 & 0 & 0 \\ & S_{xx}\left(1 - v_{xz}^2 \frac{E_{xx}}{E_{zz}}\right) & S_{xx}v_{yz} + v_{xy}v_{xz}\frac{E_{xx}}{E_{yy}} & 0 & 0 & 0 \\ & & S_{zz}\left(1 - v_{xy}^2 \frac{E_{xx}}{E_{yy}}\right) & 0 & 0 & 0 \\ & & & G_{xy} & 0 & 0 \\ & & & & G_{yz} & 0 \\ & \text{sym} & & & & G_{zx} \end{bmatrix}$$

Equation 1-46

and

$$\frac{S_{xx}}{E_{xx}} = \frac{S_{yy}}{E_{yy}} = \frac{S_{zz}}{E_{zz}} = \left(1 - v^2 \frac{E_{xx}}{E_{yy}} - v^2 \frac{E_{yy}}{E_{zz}} - v^2 \frac{E_{xx}}{E_{yy}} - 2v_{xy}v_{yz}v_{xz}\frac{E_{xx}}{E_{zz}}\right)^{-1}$$

Equation 1-47

If the material is isotropic so that

$$E_{xx} = E_{yy} = E_{zz} = E$$

Equation 1-48 a

$$v_{xy} = v_{yx} = v_{yz} = v_{zy} = v_{zx} = v_{xz} = v$$

Equation 1-48 b

$$G_{xy} = G_{yz} = G_{zx} = G = \frac{E}{2(1 + v)}$$

Equation 1-48 c

$$\alpha_{xx} = \alpha_{yy} = \alpha_{zz} = \alpha$$

Equation 1-48 d

the elastic coefficient and material property matrices become

$$F = \begin{bmatrix} \dfrac{1}{E} & -\dfrac{v}{E} & -\dfrac{v}{E} & 0 & 0 & 0 \\[2mm] & \dfrac{1}{E} & -\dfrac{v}{E} & 0 & 0 & 0 \\[2mm] & & \dfrac{1}{E} & 0 & 0 & 0 \\[2mm] & & & \dfrac{2(1+v)}{E} & 0 & 0 \\[2mm] & & & & \dfrac{2(1+v)}{E} & 0 \\[2mm] \text{sym} & & & & & \dfrac{2(1+v)}{E} \end{bmatrix}$$

Equation 1-49

$$C = \begin{bmatrix} \dfrac{(1-v)E}{(1+v)(1-2v)} & \dfrac{vE}{(1+v)(1-2v)} & \dfrac{vE}{(1+v)(1-2v)} & 0 & 0 & 0 \\[3mm] & \dfrac{(1-v)E}{(1+v)(1-2v)} & \dfrac{vE}{(1+v)(1-2v)} & 0 & 0 & 0 \\[3mm] & & \dfrac{(1-v)E}{(1+v)(1-2v)} & 0 & 0 & 0 \\[3mm] & & & \dfrac{E}{2(1-2v)} & 0 & 0 \\[3mm] & & & & \dfrac{E}{2(1-2v)} & 0 \\[3mm] \text{sym} & & & & & \dfrac{E}{2(1-2v)} \end{bmatrix}$$

Equation 1-50

and

$$C\bar{A} = \left\{ \begin{array}{c} \dfrac{E\alpha}{1-2v} \\[2mm] \dfrac{E\alpha}{1-2v} \\[2mm] \dfrac{E\alpha}{1-2v} \\[1mm] 0 \\ 0 \\ 0 \end{array} \right\}$$

Equation 1-51

1.3.2. Plane Strain

For a body in a state of plain strain, the displacements, and therefore the loading are assumed to be independent of one coordinate, say the z-coordinate, so that

$$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zx} = 0$$

Equation 1-52

Then for an orthotropic material with one of the material axes coinciding with the longitudinal axis

$$\sigma_{xz} = \sigma_{yz} = 0$$

Equation 1-53

and

$$\sigma_{zz} = v_{xz}\sigma_{xx} + v_{yz}\sigma_{yy} - \alpha_{zz}E_{zz}\Delta T$$

Equation 1-54

The remaining stresses are given in the material coordinate system by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = \begin{bmatrix} S_{xx}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right) & S_{xx}\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) & 0 \\ & S_{yy}\left(1 - v_{yy}^2 \frac{E_{xx}}{E_{zz}}\right) & 0 \\ \text{sym} & & G_{xy} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xx} \\ \varepsilon_{xx} \end{Bmatrix} - \begin{Bmatrix} \alpha_{xx} + v_{xz}\alpha_{zz} \\ \alpha_{yy} + v_{yz}\alpha_{zz} \\ 0 \end{Bmatrix}\Delta T\right)$$

Equation 1-55

If the angle between the x', y' coordinate axes and the x, y material axes are denoted by θ (positive in the counterclockwise direction), then the stress0strain relations become

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \sigma_{z'z'} \end{Bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} \\ & C'_{22} & C'_{23} \\ \text{sym} & & C'_{33} \end{bmatrix}\begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{z'z'} \end{Bmatrix} - \begin{Bmatrix} A'_{x'x'} \\ A'_{y'y'} \\ A'_{z'z'} \end{Bmatrix}\Delta T$$

Equation 1-56

with

$$\begin{Bmatrix} C'_{11} \\ C'_{22} \end{Bmatrix} = S_{xx}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right)\begin{Bmatrix} \cos^4\theta \\ \sin^4\theta \end{Bmatrix} + S_{yy}\left(1 - v_{xz}^2 \frac{E_{xx}}{E_{zz}}\right)\begin{Bmatrix} \sin^4\theta \\ \cos^4\theta \end{Bmatrix}$$

$$+ \left[2S_{xx}\left(v_{xy} - v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) + 4G_{xy}\right]\sin^2\theta\cos^2\theta$$

Equation 1-57 a

$$C'_{33} = \left[S_{xx}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right) + S_{xx}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right) - 2S_{xx}\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) - 4G_{xy} \right] \sin^2\theta \cos^2\theta + G_{xy}$$

<div align="center">**Equation 1-57 b**</div>

$$C'_{12} = \left[S_{xx}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right) + S_{yy}\left(1 - v_{xz}^2 \frac{E_{xx}}{E_{zz}}\right) - 2S_{xx}\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) - 4G_{xy} \right] \sin^2\theta \cos^2\theta$$
$$+ S_{xx}\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right)$$

<div align="center">**Equation 1-57 c**</div>

$$\left.\begin{matrix} C'_{13} \\ C'_{23} \end{matrix}\right\} = \left\{ \left[S_{xx}\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) - S_{xx}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right) + 2G_{xy} \right] \begin{cases} \cos^2\theta \\ \sin^2\theta \end{cases}\right.$$
$$\left. + \left[S_{yy}\left(1 - v_{xz}^2 \frac{E_{xx}}{E_{zz}}\right) - S_{xx}\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) - 2G_{xy} \right] \begin{cases} \sin^2\theta \\ \cos^2\theta \end{cases} \sin\theta\cos\theta \right.$$

<div align="center">**Equation 1-57 d**</div>

and

$$A'_{x'x'} = S_{xx}\left\{ \left[\alpha_{xx}v_{xz}\alpha_{zz}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right) + \left(\alpha_{yy} + v_{yz}\alpha_{zz}\right) + \left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) \right] \cos^2\theta \right.$$
$$\left. + \left(\alpha_{xx} + v_{yz}\alpha_{zz}\right)\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right)\sin^2\theta \right\} S_{yy}\left(1 - v_{xz}^2 \frac{E_{xx}}{E_{zz}}\right)\left(\alpha_{yy} + v_{yz}\alpha_{zz}\right)\sin^2\theta$$

<div align="center">**Equation 1-58 a**</div>

$$A'_{y'y'} = S_{xx}\left\{ \left[\alpha_{xx}v_{xz}\alpha_{zz}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right) + \left(\alpha_{yy} + v_{yz}\alpha_{zz}\right) + \left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right) \right] \sin^2\theta \right.$$
$$\left. + \left(\alpha_{xx} + v_{yz}\alpha_{zz}\right)\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right)\cos^2\theta \right\} S_{yy}\left(1 - v_{xz}^2 \frac{E_{xx}}{E_{zz}}\right)\left(\alpha_{yy} + v_{yz}\alpha_{zz}\right)\cos^2\theta$$

<div align="center">**Equation 1-58 b**</div>

$$A'_{x'y'} = \left\{ S_{yy}\left(1 - v_{yz}^2 \frac{E_{yy}}{E_{zz}}\right)\left(\alpha_{yy} + v_{yz}\alpha_{zz}\right) \right.$$
$$+ S_{xx}\left(v_{xy} + v_{xz}v_{yz}\frac{E_{yy}}{E_{zz}}\right)\left[\alpha_{xx} - \alpha_{yy} + \left(v_{xz} - v_{yz}\right)\alpha_{zz}\right]$$
$$\left. - S_{xx}\left(1 - v_{xz}^2 \frac{E_{xx}}{E_{zz}}\right)\left(\alpha_{xx} + v_{yz}\alpha_{zz}\right) \right\} \sin\theta\cos\theta$$

<div align="center">**Equation 1-58 c**</div>

For an isotropic material, the stress-strain relationship in any coordinate system is given by

$$
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} =
\begin{bmatrix}
\dfrac{(1-v)E}{(1+v)(1-2v)} & \dfrac{vE}{(1+v)(1-2v)} & 0 \\[2ex]
 & \dfrac{(1-v)E}{(1+v)(1-2v)} & 0 \\[2ex]
\text{sym} & & \dfrac{E}{2(1+v)}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}
- \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \dfrac{E\alpha\Delta T}{(1+v)(1-2v)}
$$

Equation 1-59

1.3.3. Plane Stress

A body in state of plane stress is characterized by the relations

$$
\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0
$$

Equation 1-60

which are generally satisfied only approximately. Then for an orthotropic material with coincident material and body z-coordinate axes, the pertinent strains in the material coordinate system are given in terms of the remaining stress components by

$$
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} =
\begin{bmatrix}
\dfrac{1}{E_{xx}} & -\dfrac{v_{xy}}{E_{yy}} & 0 \\[2ex]
 & \dfrac{1}{E_{yy}} & 0 \\[2ex]
\text{sym} & & \dfrac{1}{G_{xy}}
\end{bmatrix}
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}
+ \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \end{Bmatrix} \Delta T
$$

Equation 1-61

which can be inverted to yield

$$
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} =
\begin{bmatrix}
\dfrac{E_{xx}}{1-v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}} & \dfrac{v_{xy}E_{xx}}{1-v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}} & 0 \\[3ex]
 & & 0 \\[2ex]
\text{sym} & & G_{xy}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_{xx} - \alpha_{xx}\Delta T \\ \varepsilon_{yy} - \alpha_{yy}\Delta T \\ \varepsilon_{xy} \end{Bmatrix}
$$

Equation 1-62

If the body coordinates axes x', y' are rotated through a counterclockwise angle θ with respect to the material x, y axes, the stress strain relations are given in body coordinates by

$$\begin{Bmatrix} \sigma_{x'\,x'} \\ \sigma_{y'\,y'} \\ \sigma_{x'\,y'} \end{Bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} \\ & C'_{22} & C'_{23} \\ \text{sym} & & C'_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x'\,x'} \\ \varepsilon_{y'\,y'} \\ \varepsilon_{x'\,y'} \end{Bmatrix} - \begin{Bmatrix} A'_{11} \\ A'_{22} \\ A'_{12} \end{Bmatrix} \Delta T$$

Equation 1-63

and

$$\begin{Bmatrix} C'_{11} \\ C'_{22} \end{Bmatrix} = \frac{1}{1 - v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}} \left[E_{xx} \begin{Bmatrix} \cos^4 \theta \\ \sin^4 \theta \end{Bmatrix} + 2v_{xy} E_{xx} \sin^2 \theta \cos^2 \theta + E_{yy} \begin{Bmatrix} \sin^4 \theta \\ \cos^4 \theta \end{Bmatrix} \right] + 4G_{xy} \sin^2 \theta \cos^2 \theta$$

Equation 1-64 a

$$C'_{33} = \left[\frac{1}{1 - v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}} \left(E_{xx} + E_{yy} - 2v_{xy} E_{xx} \right) - 4G_{xy} \right] \sin^2 \theta \cos^2 \theta + G_{xy}$$

Equation 1-64 b

$$C'_{12} = \frac{v_{xy} E_{xx}}{1 - v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}} \left[\frac{1}{1 - v_{xy}^2 \dfrac{E_{xx}}{E_{y}}} \left(E_{xx} + E_{yy} - 2v_{xy} E_{xx} \right) - 4G_{xy} \right] \sin^2 \theta \cos^2 \theta$$

Equation 1-64 c

$$\begin{Bmatrix} C'_{13} \\ C'_{23} \end{Bmatrix} = \left\{ \left[\frac{1}{1 - v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}} \left(E_{yy} - v_{xy} E_{xx} \right) - 2G_{xy} \right] \begin{Bmatrix} \sin^2 \theta \\ \cos^2 \theta \end{Bmatrix} - \left[\frac{1}{1 - v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}} \left(E_{xx} - v_{xy} E_{xx} \right) - 2G_{xy} \right] \begin{Bmatrix} \cos^2 \theta \\ \sin^2 \theta \end{Bmatrix} \right\} \sin \theta \cos \theta$$

Equation 1-64 d

For an isotropic material the stress-strain relations in any orthogonal coordinate systems are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = \begin{bmatrix} \dfrac{E}{1 - v^2} & \dfrac{vE}{1 - v^2} & 0 \\ & \dfrac{E}{1 - v^2} & 0 \\ \text{sym} & & \dfrac{E}{2(1 + v)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} - \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \frac{E\alpha\Delta T}{1 + v}$$

Equation 1-65

and

$$
\begin{Bmatrix} A'_{11} \\ A'_{22} \end{Bmatrix} = \left[\left(\alpha_{xx} E_{xx} + v_{xy} \alpha_{yy} E_{xx} \right) \begin{Bmatrix} \cos^2 \theta \\ \sin^2 \theta \end{Bmatrix} + \alpha_{xx} v_{xy} E_{xx} + \alpha_{yy} E_{yy} \begin{Bmatrix} \sin^2 \theta \\ \cos^2 \theta \end{Bmatrix} \right] \frac{1}{1 - v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}}
$$

Equation 1-66 a

$$
A'_{12} = \left\{ \alpha_{yy} E_{yy} - \left[\alpha_{xx} (1 - v_{xy}) + v_{xy} \alpha_{yy} \right] E_{xx} \right\} \frac{\sin \theta \cos \theta}{1 - v_{xy}^2 \dfrac{E_{xx}}{E_{yy}}}
$$

Equation 1-66 b

1.3.4. Axisymmetric Stress State

A final special care is that of a body of revolution in a state of axisymmetric stress. In this case, a cylindrical coordinate system (Fig. 9) is used, with the displacement u_r and u_z and the temperature independent of the ϕ coordinate and u_ϕ equal to zero. The non-zero strains are then

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

Equation 1-67 a

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

Equation 1-67 b
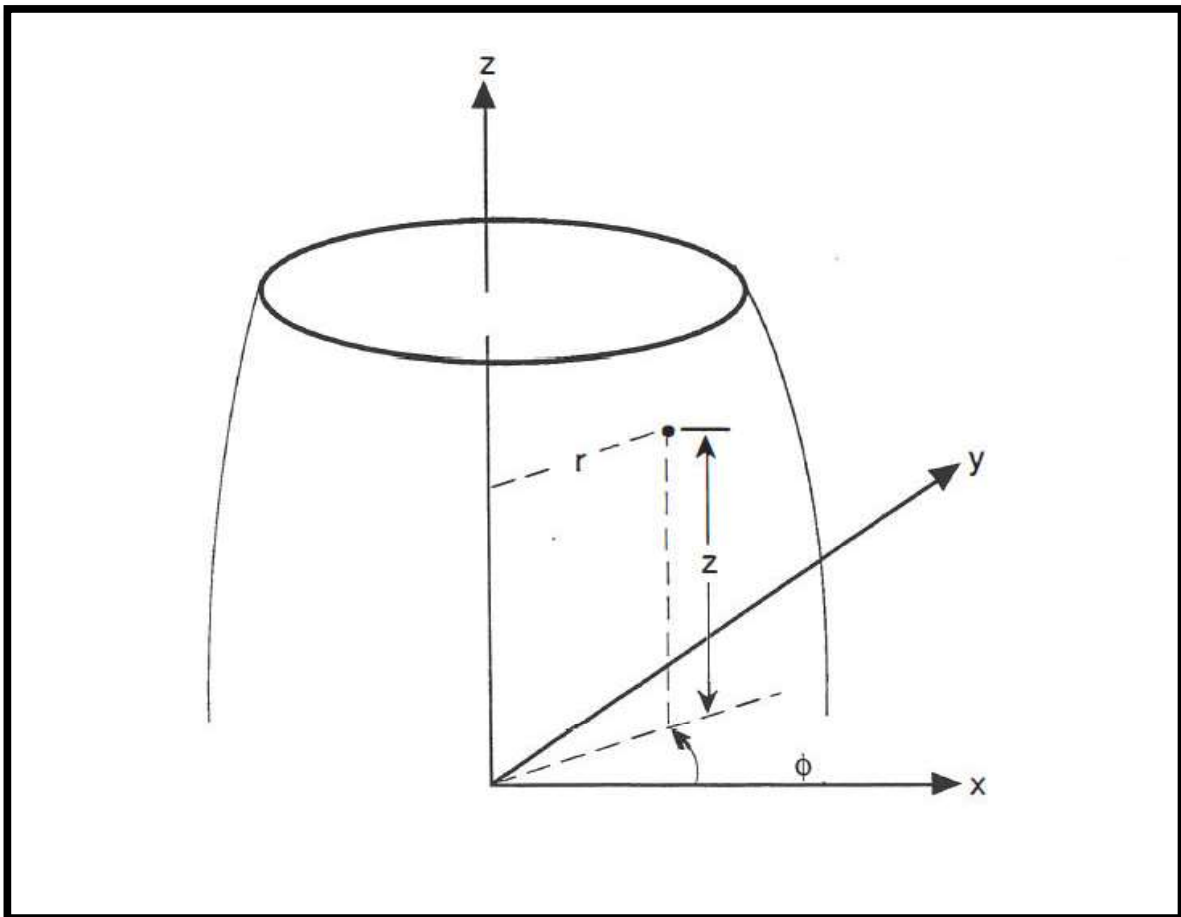
FIGURE 9 CYLINDRICAL COORDINATE SYSTEM

$$\varepsilon_{\varphi\varphi} = \frac{u_r}{r}$$

Equation 1-67 c

$$\varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

Equation 1-67 d

The material is assumed to be locally conically orthotropic. One material axis coincides with the circumferential direction. The others are oriented at angles θ and $\theta + \pi/2$, with respect to the radial direction (Fig. 10). Then shearing stresses $\sigma_{s\phi}$ and $\sigma_{n\phi}$ vanish and the relations between the remaining stresses and strains are given by
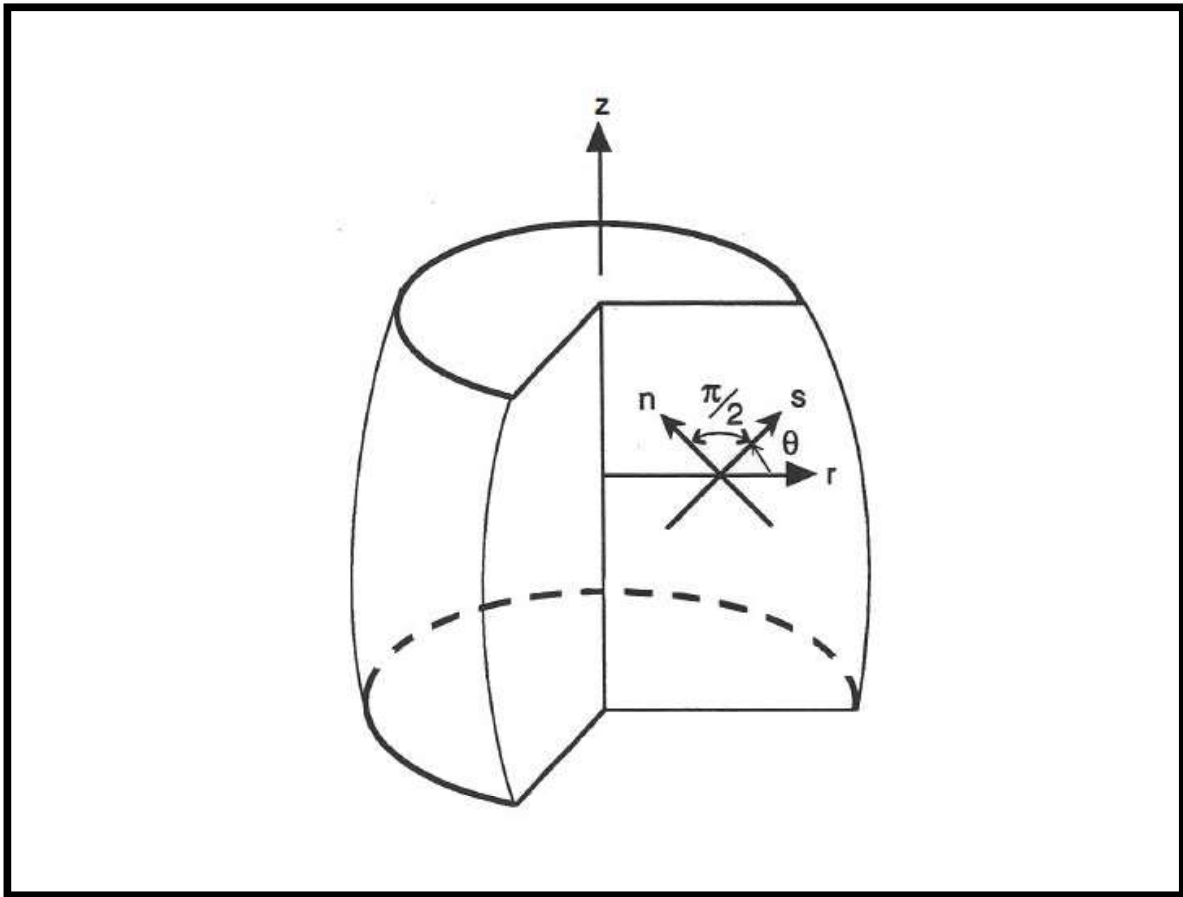
FIGURE 10   DIRECTIONS OF MATERIAL ORTHOTROPY AT A POINT IN A BODY OF REVOLUTION

$$\begin{Bmatrix} \varepsilon_{ss} \\ \varepsilon_{nn} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{sn} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E_{ss}} & -\dfrac{v_{sn}}{E_{nn}} & -\dfrac{v_{s\varphi}}{E_{\varphi\varphi}} & 0 \\[2ex] & \dfrac{1}{E_{nn}} & -\dfrac{v_{n\varphi}}{E_{\varphi\varphi}} & 0 \\[2ex] & & \dfrac{1}{E_{\varphi\varphi}} & 0 \\[2ex] & & & \dfrac{1}{G_{sn}} \end{bmatrix} \begin{Bmatrix} \sigma_{ss} \\ \sigma_{nn} \\ \sigma_{\varphi\varphi} \\ \sigma_{sn} \end{Bmatrix} + \begin{Bmatrix} \alpha_{ss} \\ \alpha_{nn} \\ \alpha_{\varphi\varphi} \\ \alpha_{sn} \end{Bmatrix} \Delta T$$

Equation 1-68

or in the inverted form

$$\begin{Bmatrix} \sigma_{ss} \\ \sigma_{nn} \\ \sigma_{\varphi\varphi} \\ \sigma_{sn} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ & C_{22} & C_{23} & 0 \\ & & C_{33} & 0 \\ \text{sym} & & & G_{sn} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_{ss} \\ \varepsilon_{nn} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{sn} \end{Bmatrix} - \begin{Bmatrix} \alpha_{ss} \\ \alpha_{nn} \\ \alpha_{\varphi\varphi} \\ 0 \end{Bmatrix} \Delta T \right)$$

Equation 1-69

with

$$C_{11} = \left(1 - v_{n\varphi}^2 \frac{E_{nn}}{E_{\varphi\varphi}} \right) S_{ss}$$

Equation 1-70 a

$$C_{12} = \left(v_{ns} + v_{s\varphi}v_{n\varphi} \frac{E_{nn}}{E_{\varphi\varphi}} \right) S_{ss}$$

Equation 1-70 b

$$C_{13} = \left(v_{s\varphi} + v_{sn}v_{n\varphi} \frac{E_{nn}}{E_{\varphi\varphi}} \right) S_{ss}$$

Equation 1-70 c

$$C_{22} = \left(1 - v_{s\varphi}^2 \frac{E_{ss}}{E_{\varphi\varphi}} \right) S_{nn}$$

Equation 1-70 d

$$C_{23} = \left(v_{n\varphi} + v_{sn}v_{s\varphi} \frac{E_{ss}}{E_{\varphi\varphi}} \right) S_{nn}$$

Equation 1-70 e

$$C_{33} = \left(1 - v_{ns}^2 \frac{E_{ss}}{E_{nn}}\right) S_{\varphi\varphi}$$

Equation 1-70 f

and

$$\frac{S_{ss}}{E_{ss}} = \frac{S_{\varphi\varphi}}{E_{\varphi\varphi}} = \frac{S_{nn}}{E_{nn}} = \left(1 - v_{s\varphi}^2 \frac{E_{ss}}{E_{\varphi\varphi}} - v_{n\varphi}^2 \frac{E_{nn}}{E_{\varphi\varphi}} - v_{sn}^2 \frac{E_{ss}}{E_{nn}} - 2v_{sn}v_{n\varphi}v_{s\varphi} \frac{E_{ss}}{E_{\varphi\varphi}}\right)^{-1}$$

Equation 1-71

In the r, z coordinate system the relations become

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\varphi\varphi} \\ \sigma_{rz} \end{Bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & C'_{14} \\ & C'_{22} & C'_{23} & C'_{24} \\ & & C'_{33} & C'_{34} \\ \text{sym} & & & C'_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{rz} \end{Bmatrix} - \begin{Bmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{Bmatrix} \Delta T$$

Equation 1-72

where

$$C'_{11} = C_{11}\cos^4\theta + C_{22}\sin^4\theta + 2(C_{12} + 2G_{sn})\sin^2\theta\cos^2\theta$$

Equation 1-73 a

$$C'_{12} = C_{12} + \left[C_{11} + C_{22} - 2(C_{12} + 2G_{sn})\right]\sin^2\theta\cos^2\theta$$

Equation 1-73 b

$$C'_{13} = C_{13}\cos^2\theta + C_{23}\sin^2\theta$$

Equation 1-73 c

$$C'_{14} = \left[C_{11}\cos^2\theta - C_{22}\sin^2\theta - (C_{12} + 2G_{sn})(\cos^2\theta - \sin^2\theta)\right]\sin\theta\cos\theta$$

Equation 1-73 d

$$C'_{22} = C_{11}\sin^4\theta + C_{22}\cos^4\theta + 2(C_{12} + 2G_{sn})\sin^2\theta\cos^2\theta$$

Equation 1-73 e

$$C'_{23} = C_{13}\sin^2\theta + C_{23}\cos^2\theta$$

Equation 1-73 f

$$C'_{24} = C_{11}\sin^2\theta - C_{22}\cos^2\theta + C_{12} + 2G_{sn}\cos^2\theta - \sin^2\theta\sin\theta\cos\theta$$

Equation 1-73 g

$$C_{33}' = C_{33}$$

Equation 1-73 h

$$C_{34}' = \left(C_{13} - C_{23}\right)\sin\theta\cos\theta$$

Equation 1-73 i

$$C_{44}' = G_{sn} + \left[C_{11} + C_{22} - 2\left(C_{12} + 2G_{sn}\right)\right]\sin^2\theta\cos^2\theta$$

Equation 1-73 j

and

$$A_1' = \left(C_{11}\alpha_{ss} + C_{12}\alpha_{nn} + C_{13}\alpha_{\varphi\varphi}\right)\cos^2\theta + \left(C_{12}\alpha_{ss} + C_{22}\alpha_{nm} + C_{23}\alpha_{\varphi\varphi}\right)\sin^2\theta$$

Equation 1-74 a

$$A_2' = \left(C_{11}\alpha_{ss} + C_{12}\alpha_{nn} + C_{13}\alpha_{\varphi\varphi}\right)\sin^2\theta + \left(C_{12}\alpha_{ss} + C_{22}\alpha_{nm} + C_{23}\alpha_{\varphi\varphi}\right)\cos^2\theta$$

Equation 1-74 b

$$A_3' = C_{13}\alpha_{ss} + C_{23}\alpha_{nm} + C_{33}\alpha_{\varphi\varphi}$$

Equation 1-74 c

$$A_4' = \left(C_{11} - C_{12}\right)\alpha_{ss} + \left(C_{12} - C_{22}\right)\alpha_{nn} + \left(C_{13} - C_{23}\right)\alpha_{\varphi\varphi}\sin\theta\cos\theta$$

Equation 1-74 d

For an isotropic material the stress-strain relations are

$$
\begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\varphi\varphi} \\ \sigma_{rz} \end{Bmatrix}
=
\begin{bmatrix}
\dfrac{(1-v)E}{(1+v)(1-2v)} & \dfrac{vE}{(1+v)(1-2v)} & \dfrac{vE}{(1+v)(1-2v)} & 0 \\[2ex]
 & \dfrac{(1-v)E}{(1+v)(1-2v)} & \dfrac{vE}{(1+v)(1-2v)} & 0 \\[2ex]
 & & \dfrac{(1-v)E}{(1+v)(1-2v)} & 0 \\[2ex]
 & & & \dfrac{E}{2(1+v)}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{rz} \end{Bmatrix}
-
\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}
\dfrac{E\alpha\Delta T}{1-2v}
$$

Equation 1-75

Chapter 2. The Finite Element Method

2.1.   The Principle of Minimum Potential Energy

A variational principle which is equivalent to the differential equations of equilibrium and boundary conditions of linear elastic solids may be derived from the principle of virtual work. This principle may be expressed mathematical as

$$\iiint\limits_{V} \sigma^T\,\delta\varepsilon dV = \iint\limits_{S} t_n^T \delta u dA + \iiint\limits_{V} f^T \delta u dV$$

Equation 2-1

where integrals are over the volume V and over the surface S of the deformed body. The left side of the equation is called the internal virtual work, the virtual work of internal forces, while the right side is the virtual work of surface and body forces. Here σ is the actual stress matrix satisfying the equations of equilibrium, t_n is the surface stress vector, f is a body force vector (force per unit volume)

$$f = \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}$$

Equation 2-2

δu is an arbitrary infinitesimal change in the displacement vector u, and

$$\delta\varepsilon = \begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \\ \delta\varepsilon_{xy} \\ \delta\varepsilon_{yz} \\ \delta\varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial\delta u_x}{\partial_x} \\[2ex] \dfrac{\partial\delta u_y}{\partial_y} \\[2ex] \dfrac{\partial\delta u_z}{\partial_z} \\[2ex] \dfrac{\partial\delta u_x}{\partial_y}+\dfrac{\partial\delta u_y}{\partial_x} \\[2ex] \dfrac{\partial\delta u_y}{\partial_z}+\dfrac{\partial\delta u_y}{\partial_x} \\[2ex] \dfrac{\partial\delta u_z}{\partial_x}+\dfrac{\partial\delta u_z}{\partial_y} \end{Bmatrix}$$

Equation 2-3

If the surface is divided into the region S_u on which displacements are prescribed and S_σ on which the stress vector is prescribed, and if the displacement matrix **u** is chosen so as to satisfy the displacement boundary conditions on S_u, the principle of virtual work may be rewritten for a linearly elastic body which satisfies Hooke's law as the principle of minimum potential energy

$$\delta\Pi = 0$$

Equation 2-4

where

$$\Pi = \iiint\limits_V \varepsilon^T C \frac{1}{2}\varepsilon - \overline{A}\Delta T - u^T f \, dV - \iint\limits_{S_\sigma} u^T \, t dS$$

Equation 2-5

The functional Π is called the potential energy of the deformed body. The integral

$$U = \frac{1}{2} \iiint\limits_V \varepsilon^T C \varepsilon dV$$

Equation 2-6

is called the strain energy of the body. The principle may be stated as follows:

Of all displacement states which satisfy boundary conditions on displacement, the unique displacement state which satisfies the equations of equilibrium and boundary conditions on stress minimizes the potential energy of the deformed body.

Variational principles in which quantities other than the displacements vary may be obtained [10]. The principle of minimum potential energy, however, is the statement most commonly used in finite element approximations.

2.2. Strain Energy Expressions for Beams, Plates and Shells

The analysis of structures which have one or two dimensions small compared to the others is usually accomplished with the help of simplifying assumptions. These are discussed below. It will be noted that in all cases the strain energy is expressible as an integral involving an integrand of the form $\varepsilon^T C\varepsilon$.

2.2.1. Straight Beams

A beam is a three dimensional structure for which the length is large compared to the width or depth of the cross-section. The reference axis defined by the line joining the centroid of each perpendicular cross-section is straight. When shear deformations are taken into account, the stretching and bending displacements are assumed to be given by the following:

Plane cross-sections perpendicular to the undeformed centroidal axis remain plane and unchanged in shape but translate and rotate with respect to the plane normal to the deformed centroidal axis. Then for small deformations the deflection can be shown to be of the form

$$u_y = u_y(x)$$

Equation 2-7a

$$u_z = u_z(x)$$

Equation 2-7b

$$u_x = u_{x_0}(x) - y\Psi_z(x) + z\Psi_y(x)$$

Equation 2-7c

where Ψ_y and Ψ_z are rotations of the cross-section about the y and z axes, respectively. In addition, torsional deformations are assumed to be governed by St. Venant torsion theory wherein the cross-sections rotate as rigid bodies about the x-axis through an angle which varies linearly with distance from the origin and undergo axial displacements which are a functional only of position in the cross-section. Then

$$u_x = w(y, z)$$

Equation 2-8a

$$u_y = -xz\frac{d\Psi_x}{dx}$$

Equation 2-8b

$$u_z = xy\frac{d\Psi_x}{dx}$$

Equation 2-8c

where w is the cross-sectional warping function and $\frac{d\Psi_x}{dx}$ is the constant rate of change of the angle of rotation about the x-axis. Then the non-zero strains are given by

$$\varepsilon_{xx} = \frac{du_{x_0}}{dx} - y\frac{d\Psi_z}{dx} + z\frac{d\Psi_y}{dx} - \alpha\Delta T$$

Equation 2-9a

$$\varepsilon_{xy} = \left(\frac{du_y}{dx} - \Psi_z\right) + \left(\frac{\partial w}{\partial y} + y\frac{d\Psi_x}{dx}\right)$$

Equation 2-9b

$$\varepsilon_{xz} = \left(\frac{du_z}{dx} - \Psi_y\right) + \left(\frac{\partial w}{\partial z} + y\frac{d\Psi_x}{dx}\right)$$

Equation 2-9c

With the additional assumption that the direct stresses σ_{yy} and σ_{zz} are negligible compared to σ_{xx}, stress resultants are defined in terms of deformations by

$$P_x = \int_A \sigma_{xx}dA = E\left[A\frac{du_{x_0}}{dx} - \alpha\int_A \Delta TdA\right]$$

Equation 2-10a

$$M_y = \int_A z\sigma_{xx}dA = E\left(I_{yy}\frac{d\Psi_y}{dx} - I_{xy}\frac{d\Psi_z}{dx} - \alpha\int_A z\Delta TdA\right)$$

Equation 2-10b

$$M_z = -\int_A y\sigma_{xx}dA = E\left(I_{zz}\frac{d\Psi_z}{dx} - I_{xy}\frac{d\Psi_y}{dx} - \alpha\int_A y\Delta TdA\right)$$

Equation 2-10c

$$V_y = -\int_A \sigma_{xy}dA = \overline{k_y}\ GA\left(\Psi_z - \frac{du_y}{dx}\right)$$

Equation 2-10d

$$V_z = \int_A \sigma_{xz}dA = k_z\ GA\left(\Psi_y + \frac{du_z}{dx}\right)$$

Equation 2-10e

$$T = \int_A \left(y\sigma_{xz} - z\sigma_{xy}\right)dA = GJ\frac{d\Psi_x}{dx}$$

Equation 2-10f

These may be expressed in matrix form as

$$\bar{p} = \bar{C}\bar{e} - \bar{\bar{A}}$$

Equation 2-11

with

$$\bar{p}^T = \{V_x \ V_y \ V_z \ T M_y \ M_z\}$$

Equation 2-12a

$$\bar{e}^T = \left\{ \frac{du_{x_0}}{dx} \ \left(\frac{du_y}{dx} - \Psi_z \right) \ \left(\frac{du_z}{dx} + \Psi_y \right) \ \frac{d\Psi_x}{dx} \ \frac{d\Psi_y}{dx} \ \frac{d\Psi_z}{dx} \right\}$$

Equation 2-12b

$$\bar{C} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ & \bar{k}_y \, GA & 0 & 0 & 0 & 0 \\ & & \bar{k}_z \, GA & 0 & 0 & 0 \\ & & & GJ & 0 & 0 \\ & & & & EI_{zz} & -EI_{yz} \\ \text{sym} & & & & & EI_{yy} \end{bmatrix}$$

Equation 2-12c

$$\bar{\bar{A}}^T = E\alpha \left\{ \int_A \Delta T \, dA \quad 0 \quad 0 \quad 0 \quad \int_A z\Delta T \, dA \quad -\int_A y\Delta T \, dA \right\}$$

Equation 2-12d

Here P_x is the axial force, while V_y and V_z are cross-sectional shearing forces. Bending moments about the y and z axes are denoted by M_y and M_z, respectively and T is the torsional moment about the x-axis. Positive directions for these quantities are shown in Figure 11. Geometric properties are denoted by A, the cross-sectional area, and l_{yy}. L_{zz} and I_{yz}, the cross-sectional moments and product of inertia:
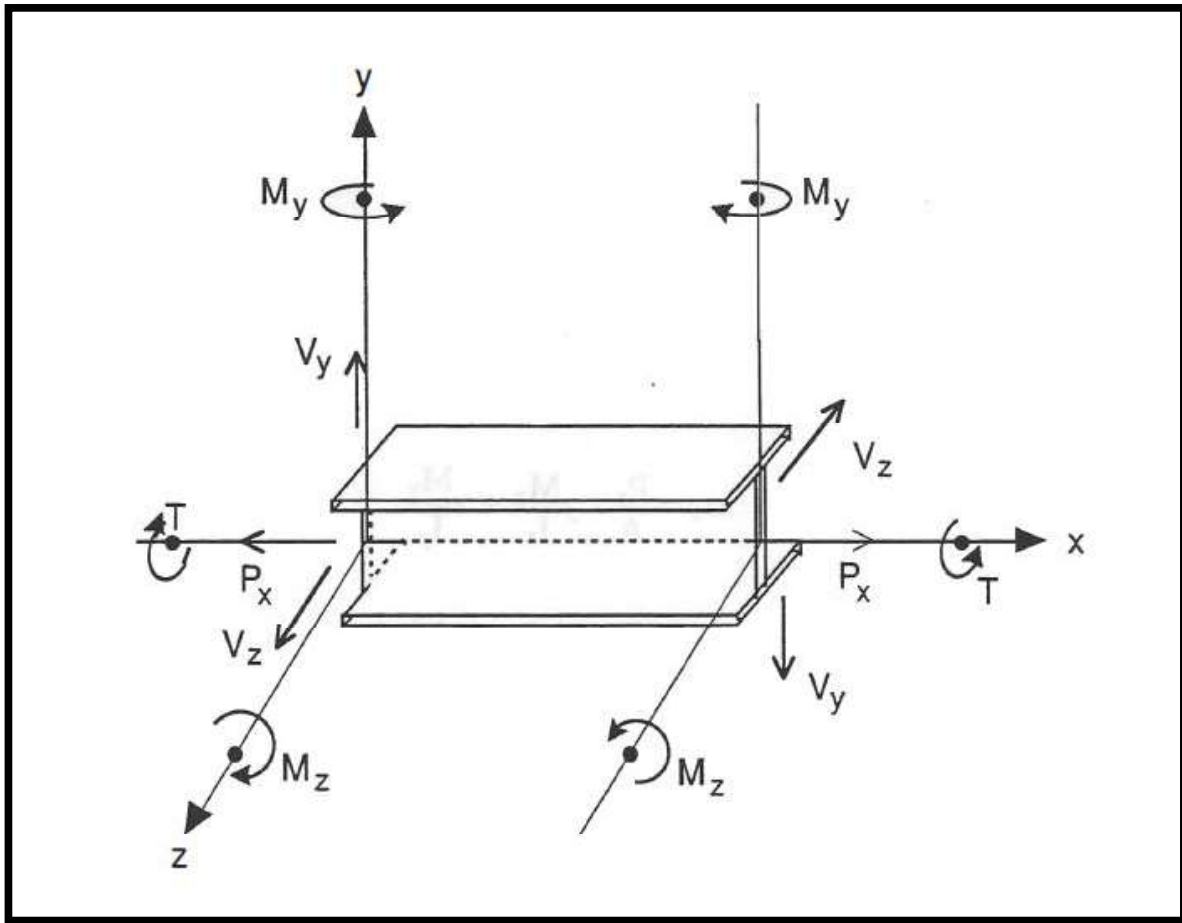
FIGURE 11   POSITIVE DIRECTIONS FOR STRESS RESULTANTS

$$I_{yy} = \int_A z^2 \, dA$$

Equation 2-13a

$$I_{zz} = \int_A y^2 \, dA$$

Equation 2-13b

$$I_{yz} = \int_A yz \, dA$$

Equation 2-13c

For principal axes, I_{yz} vanishes. J is the cross-sectional torsion constant and has no simple geometric interpretation except for solid or annular circular cross-sections for which it is the polar moment of inertia. The constants k_y and k_z are correction factors introduced to account for the actual non-uniform distribution of shearing stresses and strains over the beam cross-section

The strain energy stores in the beam is now given by

$$U = \frac{1}{2} \int_0^L \bar{e}^T \bar{C} \bar{e} \, dx$$

Equation 2-14

Axial stresses in the beam may be calculated from the expression

$$\sigma_x = \frac{P_x}{A} - y\frac{M_z}{I_z} + z\frac{M_y}{I_y}$$

Equation 2-15

The maximum shearing stress due to torsion is calculated from the formula

$$\tau_{max} = \frac{TC_{tor}}{J}$$

Equation 2-16

Where C_{tor} is a constant which depends on the shape of the cross-section [1].

2.2.2. Flat Plates

A plate is a structure for which one of the dimensions, the thickness, is small compared to the length and width. A reference plane is situated midway between the top and bottom plate surfaces. Deformations are given by assumptions which are an extension of those for beams:

Straight lines initially perpendicular to the undeformed plate middle surface remain straight and unchanged in length but rotate with respect to the normal to the deformed middle surface.

The displacement components can then be shown to be given by

$$u_x = u(x, y) + z\Psi_y \, (x, y)$$

Equation 2-17a

$$u_y = v(x, y) - z\Psi_x \, (x, y)$$

Equation 2-17b

$$u_z = w(x, y)$$

Equation 2-17c

where Ψ_x and Ψ_y are rotations about the x and y axes, respectively. The strains are now found to be

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + z\frac{\partial \Psi_y}{\partial x}$$

Equation 2-18a

$$\varepsilon_{yy} = \frac{\partial v}{\partial x} - z\frac{\partial \Psi_x}{\partial y}$$

Equation 2-18b

$$\varepsilon_{zz} = 0$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \Psi_y}{\partial y} - \frac{\partial \Psi_x}{\partial x}\right)$$

Equation 2-18c

$$\varepsilon_{xz} = \frac{\partial w}{\partial x} + \Psi_y$$

Equation 2-18d

$$\varepsilon_{yz} = \frac{\partial w}{\partial x} - \Psi_x$$

Equation 2-18e

With the direct stress σ_{zz} assumed to be negligible, the strain energy may be derived for an isotropic plate as

$$U = \frac{1}{2}\iint\limits_{A} \tilde{e}^T \ \tilde{C}\tilde{e} \ dxdy$$

Equation 2-19

where

$$\tilde{e}^T = \left\{\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \frac{\partial \Psi_y}{\partial y} \quad -\frac{\partial \Psi_x}{\partial y} \quad \left(\frac{\partial \Psi_y}{\partial y} - \frac{\partial \Psi_x}{\partial x}\right) \quad \left(\frac{\partial w}{\partial x} - \Psi_y \ \right) \quad \left(\frac{\partial w}{\partial y} + \Psi_x \ \right)\right\}$$

Equation 2-20

$$\tilde{C} = \begin{bmatrix} K & vK & 0 & 0 & 0 & 0 & 0 & 0 \\ & K & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \dfrac{1-v}{2}K & 0 & 0 & 0 & 0 & 0 \\ & & & D & vD & 0 & 0 & 0 \\ & & & & D & 0 & 0 & 0 \\ & & & & & \dfrac{1-v}{2}D & 0 & 0 \\ & & & & & & \dfrac{5}{6}Gh & 0 \\ \text{sym} & & & & & & & \dfrac{5}{6}Gh \end{bmatrix}$$

Equation 2-21

and

$$K = \frac{Eh}{1 - v^2}$$

Equation 2-22a

$$D = \frac{Eh^3}{12(1 - v^2)}$$

Equation 2-22b

The integration is over the middle surface area of the plate. The factor 5/6 is inserted into the transverse shearing strain expression to account for the nonuniform distribution of shear stress and strain over the plate thickness.

If transverse shearing strains are neglected, the normal to the undeformed plate middle surface remains normal to the deformed middle surface. Then

$$\Psi_y = -\frac{\partial w}{\partial x}$$

Equation 2-23a

$$\Psi_x = -\frac{\partial w}{\partial y}$$

Equation 2-23b

and the non-zero strains become

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$

Equation 2-24a

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} - z \frac{\partial^2 w}{\partial y^2}$$

Equation 2-24b

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w}{\partial x \, \partial y}$$

Equation 2-24c

The strain matrix $\tilde{e}$ and $\tilde{C}$ are then replaced in Eq. (2.20) by

$$\tilde{e}^t = \left\{ \frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad \frac{\partial^2 w}{\partial x \, \partial y} \right\}$$

Equation 2-25

$$\tilde{C} = \begin{bmatrix} K & vK & 0 & 0 & 0 & 0 \\ & K & 0 & 0 & 0 & 0 \\ & & \dfrac{1-v}{2}K & 0 & 0 & 0 \\ & & & D & vD & 0 \\ & & & & D & 0 \\ \text{sym} & & & & & 2(1-v)D \end{bmatrix}$$

Equation 2-26

2.3.   The Finite Element Method

In the implementation of the principle of minimum potential energy as the basis of approximate solutions by means of the finite element method of analysis, the region under consideration is divided into a finite number of subregions, say N, called "elements" (Fig. 12).
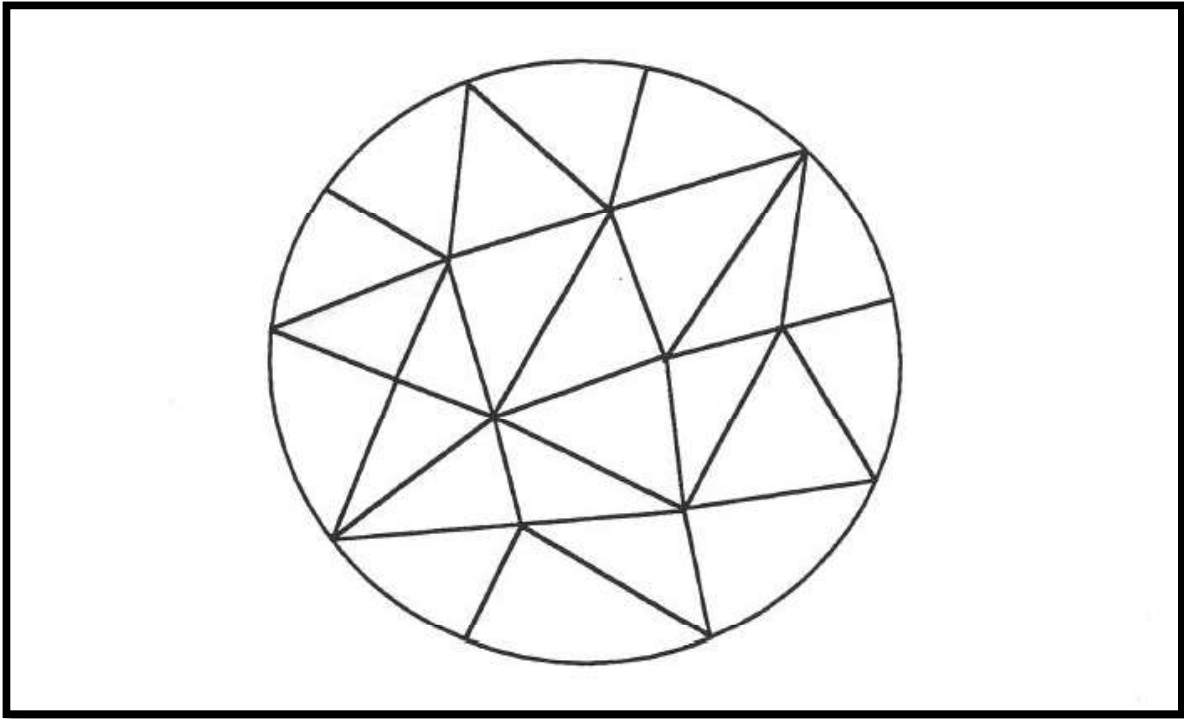
FIGURE 12   DIVISION OF A REGION INTO SUBREGIONS

The total potential energy of the region is then the sum of the potential energy of the subregions, i.e.

$$\pi = \sum_{i=1}^{N} \pi_i$$

Equation 2-27

with

$$\pi_i = \int_{V_i} \left[\varepsilon^T C \left(\frac{1}{2} \varepsilon - \overline{A} \Delta T \right) - f^T u \right] dV - \int_{S_{\sigma_i}} t^T u dS$$

Equation 2-28

The volume V_i is that of the ith element while S_{σ_i} is that portion of the surface which bounds the ith element.

The displacement matrix **u** is now represented within a typical element as

$$u_i = D_i \; q_i$$

Equation 2-29

where the components of $\mathbf{q_i}$ are displacements and possibly displacement derivatives at a number of nodal points of the element and those of $\mathbf{D_i}$ are functions of position within the element, called interpolation functions, which define the variation of the displacement matrix within the element and on its surface. Since the displacement matrix **u** must be continuous over the entire region, it follows that the displacements at the common nodes of the interelement boundary of two adjoining elements must be the same and that the functional representations of the displacements over the common · boundary must be identical. The strain matrix ε_i is then obtained as

$$\varepsilon_i = B_i \; q_i$$

Equation 2-30

in which B_i is a matrix the elements of which are, in general, function of position. Thus

$$\pi_i = q_i^T \left(\frac{1}{2} k_i \; q_i \; - F_i \; \right)$$

Equation 2-31

with

$$k_i = \int_{V_i} B_i^T C_i \; B_i \; dV$$

Equation 2-32

and

$$F_i = \int_{v_i} D_i^T f dV + \int_{S_{\sigma_i}} D_i^T t_n \, dS + \int_{v_i} B_i^T C_i \, \overline{A_i} \, \Delta T_i \, dV$$

Equation 2-33

The matrix k_i is called the element stiffness matrix. The components of F_i are equivalent applied nodal forces which are consistent with the assumed displacement distribution.

With some manipulation, the potential energy of the entire region given by

$$\pi = \sum_{i=1}^{N} \pi_i$$

Equation 2-34

may be expressed in the form

$$\pi = q^T \left(\frac{1}{2} Kq - F \right)$$

Equation 2-35

where q is the matrix of all nodal displacements and derivatives arranged consecutively, K is the assembled symmetric stiffness matrix of the entire region and F is the assembled nodal load matrix. The relationship between q_i and q may be defined by

$$q_i = M_i \, q$$

Equation 2-36

where M_i is a matrix giving the identification between nodal displacements q_i of region I and the elements of the total nodal displacement matrix q. Then the stiffness matrix K is given by

$$K = \sum_{i=1}^{N} M_i^T k_i \, M$$

Equation 2-37a

and

$$F = \sum_{i=1}^{N} M_i^T F_i$$

Equation 2-37b

The potential energy must now be minimized with respect to each of the unknown nodal displacements and derivatives, say **M** in number, while those nodal displacements on the surface S_u must satisfy the prescribed displacement conditions. Then

$$\delta\pi = \sum_{j=1}^{M} \frac{\partial\pi}{\partial q_j}\,\delta q_i \ = 0$$

Equation 2-38

where the summation is over all unknown values of q_j. Since the values of δq_j are arbitrary, the minimization procedure leads to the set of equations

$$\frac{\partial\pi}{\partial q_j} = 0 \qquad j = 1, 2, \dots M$$

Equation 2-39

where there are as many equations as there are unknown values of q_j.

The procedure outlined above leads to nodal values which, in general, are an approximation of the actual nodal values and which define approximate element stresses obtained from the equation

$$\sigma_i \ = C_i \ B_i \ q_i$$

Equation 2-40

The accuracy of the approximation may be improved by

a) Decreasing the size of the subregions and increasing their number, with the interpolation functions for each region unchanged (the h-method).
b) Increasing the number of nodal points and the complexity of the polynomial interpolation functions in the subregions, with the number of subregions unchanged (the p-method).
c) A combination of methods (a) and (b) wherein the size of some elements is decreased and their number increased with no change in the interpolation functions while for other elements the size is unchanged but the complexity of the interpolation functions and number of nodal points is increased (the h-p method).

2.4.   Interpolation Functions

The interpolation functions which define element displacements at points other than at the nodes are not completely arbitrary but are required to satisfy certain conditions imposed by the form of the strain energy function and by convergence requirements:

a) Nodal displacements consistent with constant strain should not yield nonconstant strains in the element. Nodal displacements consistent with rigid body motion should yield zero element strains.

b) The derivation of section 2.3 requires that the displacements along the common edge of adjoining elements are such that the stresses or forces along that edge do no work in acting through the virtual displacements associated with each element. For strains involving only first derivatives of displacements, the implication is that displacements along the common edge of adjoining elements, and hence the functions defining those displacements, should be identical; for strains involving second derivatives (beams, plates, shells), first derivatives of displacements should be identical as well. In particular the derivatives normal to the common edge should be identical in this case.

Elements involving displacement functions satisfying these conditions are called conforming elements.

The satisfaction of condition (b) may be difficult to achieve. It is possible, however, to obtain convergence of the finite element process with the use of displacements functions which violate continuity requirements, but which satisfy continuity in the limit as the size of the element decreases. Such elements are called non-conforming elements. The condition is ensured if the previous constant strain requirement (a) is satisfied and if displacement continuity occurs under a constant strain condition. A test for the achievement of such continuity is known as the "patch test". It requires that an arbitrary group of elements having a common node be given nodal displacements corresponding to a constant strain condition. The finite element equilibrium equation at that node must then be satisfied identically to ensure continuity satisfaction.

The interpolation functions are usually taken as polynomials of orders depending on the number of nodes and nodal variables. The coefficients of the polynomial terms are equal in number to the total number of nodal variables and are obtained by requiring that the function give the desired nodal variables at the chosen nodal points.

Linear interpolation functions yield the simplest elements and are often used. For a normalized square, for example, displacements are expressible as (Fig. 13)

$$u = \sum_{i=1}^{4} N_i \; u_i$$

Equation 2-41a

$$v = \sum_{i=1}^{4} N_i \ v_i$$

Equation 2-41b

with

$$N_1 = (1 - \xi)(1 - \eta)$$
$$N_2 = (1 - \xi)(1 - \eta)$$
$$N_3 = (1 - \xi)(1 + \eta)$$
$$N_4 = (1 - \xi)(1 + \eta)$$

$$-1 \le \xi \le +1$$
$$-1 \le \eta \le +1$$

Equation 2-42

a. Normalized Square Region

b. Displacement Variation
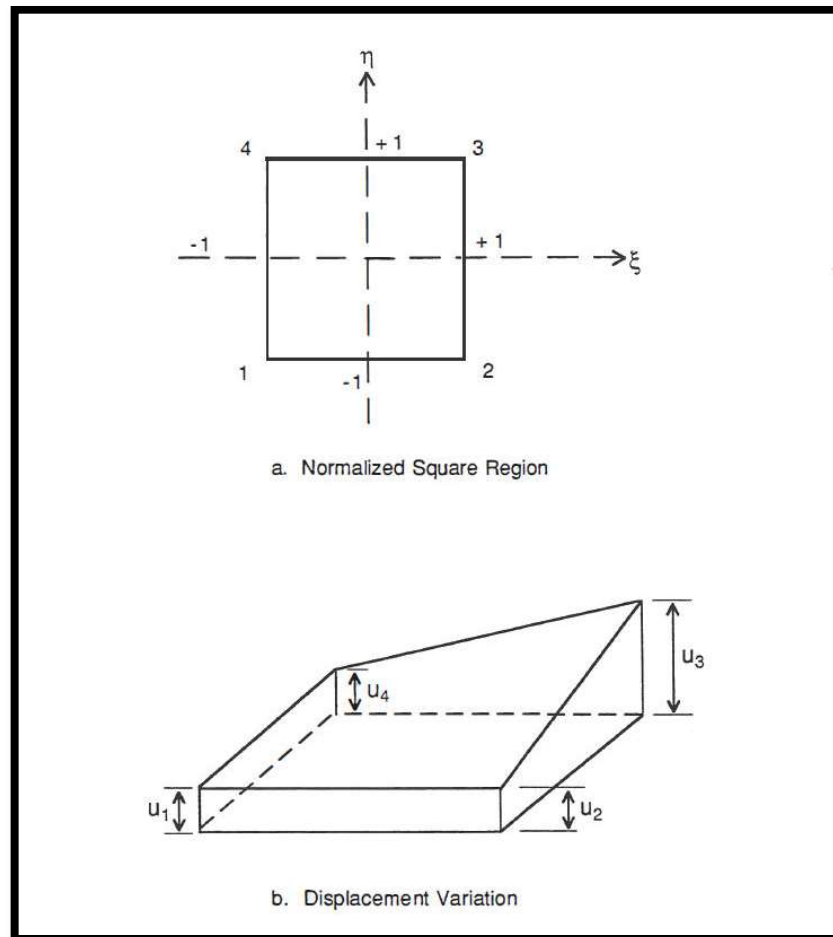
FIGURE 13   LINEAR INTERPOLATION FUNCTIONS

In the p-method of analysis considerably increased accuracy relative to the number of additional unknowns is achieved by increasing the complexity of the polynomial interpolation function while keeping the element size constant. Additional unknown quantities associated with the element can be defined in a variety of ways. For problems which require only displacement continuity (called C^0 continuity) displacements at additional nodal points along the element sides and in the element interior may be introduced. It is also possible to use higher order derivatives at the original nodes as additional unknowns, in which case continuity of first derivatives as well as displacements (called C^1 continuity) is obtainable.

In both of the above types of higher order elements, the stiffness matrix must be recalculated anew for each new set of unknowns. Hierarchic interpolation functions have the advantage of requiring only the calculation of additional row and column terms for the added unknowns. For these interpolation functions, nodes are introduced at element corners. Unknown quantities for C^0 continuity are chosen as corner node displacements and as higher order derivatives at the element side midpoints. Each additional set of derivatives is associated with polynomial terms having the same order as the derivatives and which vanish at the element corners. Hierarchical shape functions for C^1 continuity can be obtained but with more difficulty.

2.5.   Isoparametric Elements

Element interpolation functions are readily derived for single geometric shapes, i.e., triangles, rectangles, tetrahedrons, and cubes. In many applications of the finite element method it may be desirable, however, to consider elements with more irregular shapes and with curved rather than straight edges. In particular, curved elements may be used to closely model curves or surfaces. Interpolation functions for simple shapes may be extended to these more complicated shapes by a transformation of coordinates which map the boundaries of the irregular element onto those of the regular element. While this transformation can be effected in many ways, a very useful mapping is one for which the mapping function and the interpolation function are of the same form. The elements resulting from this type of mapping are called isoparametric elements. If the simple element is conforming, the isoparametric element will likewise be conforming.

Displacements in the simple geometric shape are assumed in the form

$$u(\xi) = D(\xi)q$$

Equation 2-43

where q is the matrix of nodal unknowns and coordinates ξ denote non-dimensional position in the element. These can be normalized area and volume coordinates for triangles and tetrahedra, respectively, and normalized Cartesian coordinates in rectangles and cubes. The transformation mapping the complex element onto the simple element is then taken as

$$x = D(\xi)\bar{x}$$

Equation 2-44

with $\bar{x}$ the matrix of nodal coordinates. Thus the number of nodes that must be considered on each edge is governed by the shape of the element to be mapped. For example, a quadrilateral can be obtained by using a linear interpolation function for a square and the four corner nodes. The inclusion of additional nodes and interpolation functions of higher order will result in quadrilaterals with curved sides (Figure 14).
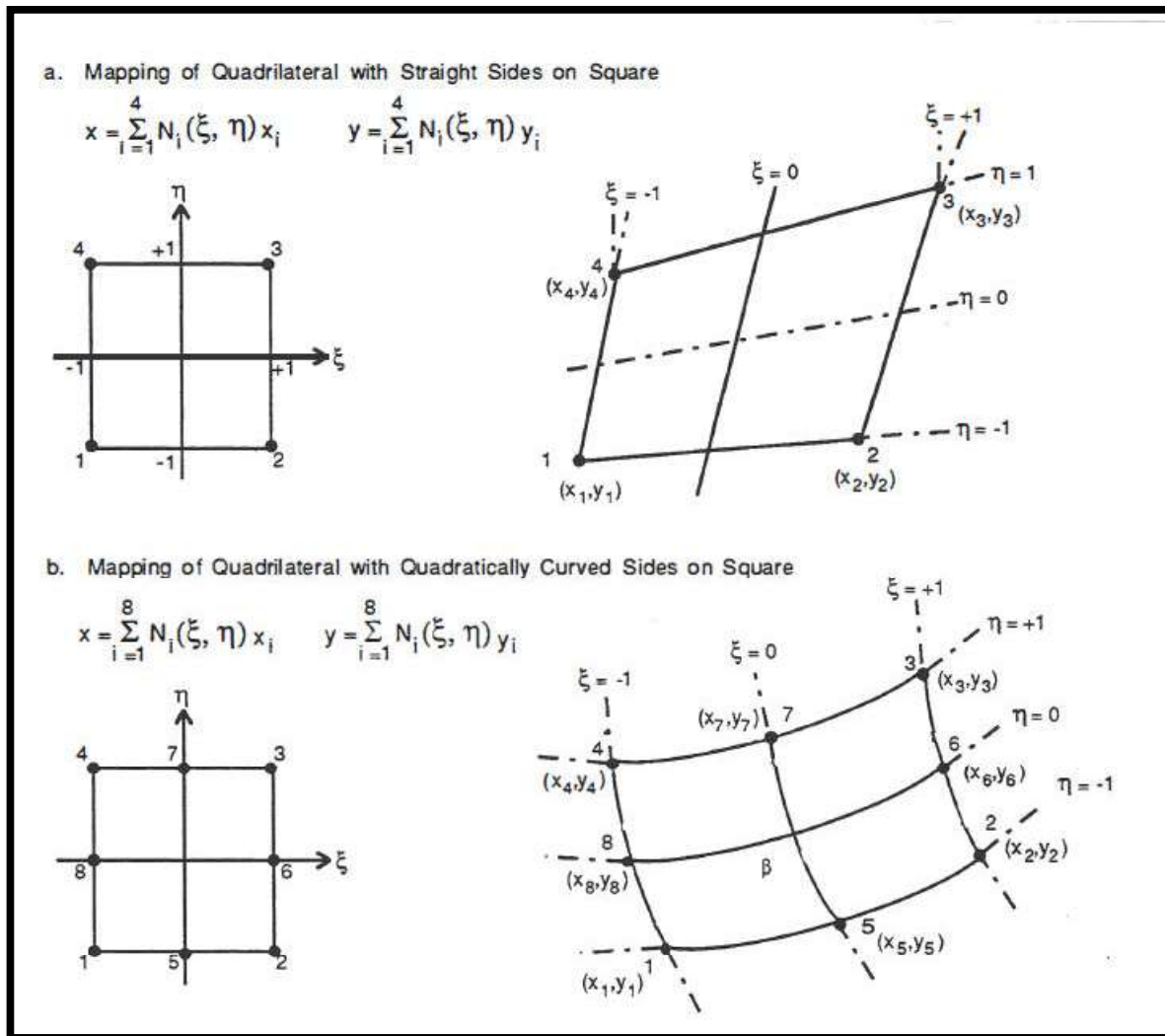
FIGURE 14   ISOPARAMETRIC MAPPINGS OF QUADRILATERAL REGIONS

In order to use isoparametric elements, the variational functions which are expressed in terms of a Cartesian coordinate system must be expressed in terms of the parametric coordinates. In two dimensions

$$x = x(\xi, \eta)$$

Equation 2-45a

$$y = y(\xi, \eta)$$

Equation 2-45b

so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}$$

Equation 2-46a

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y}$$

Equation 2-46b

where

$$\begin{Bmatrix} \dfrac{\partial \xi}{\partial x} \\ \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \eta}{\partial y} \end{Bmatrix} = \frac{1}{J}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial y}{\partial \eta} \end{Bmatrix}$$

Equation 2-47

and

$$J = \det \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}$$

Equation 2-48

Thus

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{J}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} & 0 & 0 \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} & 0 & 0 \\ 0 & 1 & \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ 0 & 0 & -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial y}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

Equation 2-49

And

$$dxdy = Jd\xi d\eta$$

Equation 2-50

The resulting integrals must be evaluated numerically since exact integration is usually difficult or impossible in terms of known functions.

It is also possible to define elements with different interpolation functions and mapping functions, i.e.,

$$u(\xi) = D(\xi)q$$

Equation 2-51a

$$x = \overline{D}(\xi)\overline{x}$$

Equation 2-51b

with

$$\overline{D}(\xi) \neq D(\xi)$$

Equation 2-51c

If the interpolation functions are of higher order than the mapping functions, the element is said to be subparametric. Hierarchic elements are subparametric if a linear or quadratic mapping transformation is used. If the interpolation functions are of lower order than the mapping functions, the element is said to be superparametric. Subparametric elements generally satisfy convergence and completeness requirements. Superparametric elements may cause problems, however, and must be investigated for completeness and compatibility.

2.6.   Numerical Integration

While the integrals required for stiffness and nodal force matrices may often be expressed in explicit form, it is sometimes more convenient and less time consuming to use methods of numerical integration for their calculation. These consist of expressing the integral as a summation of products of values of the function at specified sampling points and weighting constants.

In one dimension, then,

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{M} W_i \ f(\xi_i)$$

Equation 2-52

If the values of ξ_i are equally spaced, m sampling points yield m unknown values of Wi which can be chosen to integrate a polynomial of degree m-1 exactly. This method is known as Newton-Cotes quadrature. If, however, the locations of the sampling points are unknown as well, m sampling points yield 2m unknowns which can integrate a polynomial of 2m-1 exactly. This method is known as Gauss quadrature and is preferable in that fewer sampling points are required for a polynomial of a given degree. Consider, for example, a third degree polynomial

$$f = a + b\xi + c\xi^2 + d\xi^3 \qquad -1 < \xi < -1$$

Equation 2-53

Then

$$\int_{-1}^{1} f \, d\xi = 2a + \frac{2}{3}c$$

Equation 2-54

The use of two Guass points at $\xi=\xi_1, \xi_2$ then requires that

$$W_1 \left(a + b\xi_1 + c\xi_1^2 + d\xi_1^3\right) + W_2 \left(a + b\xi_2 + c\xi_2^2 + \xi_2^3\right) = 2a + \frac{2}{3}c$$

Equation 2-55a

Equating coefficients of a, b, c and d on both sides of the equations yields four relations from which is obtained

$$W_1 = W_2 = 1$$

Equation 2-55b

$$\xi_1 = -\xi_2 = \frac{1}{\sqrt{3}}$$

Equation 2-55c

Newton Cotes integration would have required the use of 4 sampling points.

For a square or a cube, integration can be considered to be carried out first in one direction and then in the other. Thus for a two-dimensional square

$$\int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^{M} W_i \int_{-1}^{1} f(\xi_i, \eta) d\eta = \sum_{i=1}^{M} \sum_{j=1}^{N} W_i \, W_j \, f(\xi_i, \eta_j)$$

Equation 2-56

where the point locations and weighting functions are identical to those of one dimensional Gauss quadrature. The number of sampling points need not be the same in both directions but are usually taken to be identical. For a cube, similarly,

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f\xi, \eta, \zeta \, d\xi \, d\eta \, d\zeta = \sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{P} W_i \ W_j \ W_k \ f(\xi_i \ , \eta_j \ , \zeta_k)$$

Equation 2-57

2.7.   Reduced Integration

While numerical integration using an appropriate number of Gauss sampling points can exactly integrate polynomial expressions of a given order, exact integration may lead to erroneous results in some cases. Examples of problems for which errors will occur with exact integration are those involving isotropic elasticity elements with Poisson's ratio v near 1/2 or thin beam, or plate and shell elements with shear flexibility. The strain energy associated with volume change should become small compared to to strain energy of shape change as Poisson's ratio approaches 1/2, in the former case. In the latter case, the strain energy associated with shear deformation should become small compared to the strain energy of bending as the thickness decreases.

The difficulty arises from the circumstance that the part of the structural stiffness matrix associated with the vanishing portion of the strain energy actually becomes increasingly dominant and the structure overly stiff. The displacements obtained from the analysis then decrease to zero as Poisson's ratio approaches 1/2 or thickness approaches zero, giving a set of erroneous displacements which satisfies the condition of zero volume change or of zero shearing deformation. The situation is usually remedied by reduced integration, i.e., the use of fewer Gauss sampling points for numerical integration of the offending terms than are required for exact integration. If the number of points is reduced sufficiently, that pan of the stiffness matrix will become singular and will result in accurate solutions.

2.8.   Solution of simultaneous Linear Expressions

The set of finite element equilibrium equations, however obtained, must b e solved to achieve the purpose of the analysis. Although there are a number of ways to solve simultaneous linear equations, the method which is most widely used is that of Gauss elimination. In this method the equation

$$Kq = F$$

Equation 2-58a

is transformed to the form

$$Uq = \overline{F}$$

Equation 2-58b

where U is an upper triangular matrix, i.e., all elements of the matrix below the principal diagonal are zero. The process of matrix transformation to upper diagonal form consists of using the first equation to eliminate q_1 from the second and succeeding equations. The second equation is then used to eliminate q_z from the third and succeeding equations. The process of elimination is continued until the last equation consisting of a single term in the U matrix is obtained. The values of q are then obtained by solving the last equation for qN alone, then substituting the result in the preceding equation to yield q_{N-1}, and so on.

2.9.   Stress Calculations

The end result of the analysis, the distribution of stresses in the structure, can be obtained from an appropriate finite element expression once the displacements are calculated. The stress values will vary over the element. At the boundary between adjoining elements with only C^0 continuity imposed, the first derivatives of displacement normal to the edge and henc the stresses will be discontinuous. Similarly for beams, plates and shells which require C^1 continuity, the second and mixed derivatives of displacements will usually be discontinuous at the boundary of adjoining elements and will lead to discontinuous stresses. The question of what are accurate stress values therefore arises. Investigations have shown that the most accurate stress values are those at the Gauss integration points. These values are calculated in the analysis and are extrapolated to yield stresses at element boundaries.

Chapter 3. Vibration Frequencies of Structures

3.1. Vibration Modes and Frequencies

A structure which is initially disturbed from a rest state will continue in motion without the application of force. For small deformations, this motion can be expressed as the superposition of vibration modes, each of which has a sinusoidal time variation with a distinct frequency. Such motions are called free harmonic vibrations. The modes of vibration are orthogonal, a fact which renders them useful in solving problems of the response of structures under time dependent loading as well as under specified initial conditions.

3.2. Finite Element Analysis

The variational principle for determination of vibration modes and frequencies is given by

$$\delta(U + V) = 0$$

Equation 3-1

where U is the strain energy stored in the body given by

$$U = \frac{1}{2} \iiint_V \varepsilon^T C \varepsilon \, dv$$

Equation 3-2

And V is the kinetic energy of the body

$$V = \frac{1}{2} \iiint_V \rho \left[\left(\frac{\partial u_x}{\partial t}\right)^2 + \left(\frac{\partial u_y}{\partial t}\right)^2 + \left(\frac{\partial u_z}{\partial t}\right)^2 \right] dv$$

Equation 3-3

For plates and shells where the displacements are assumed to be given by

$$u_x = \overline{u_x}\,(x, y, t) + z\Psi_y\,(x, y, t)$$

$$u_y = \overline{u_y}\,(x, y, t) - z\Psi_x\,(x, y, t)$$

$$u_z = \overline{u_z}\,(x, y, t)$$

the kinetic energy becomes

$$V = \frac{1}{2} \iint_A \left\{ \rho \left(\frac{\partial \overline{u}_x}{\partial t}\right)^2 + \left(\frac{\partial \overline{u}_y}{\partial t}\right)^2 + \left(\frac{\partial \overline{u}_z}{\partial t}\right)^2 + \frac{\rho h^3}{12}\left[\left(\frac{\partial \Psi_x}{\partial t}\right)^2 + \left(\frac{\partial \Psi_y}{\partial t}\right)^2 \right] \right\} dA$$

where integration is over the area of the middle surface.

The displacement are expressed as the product of a function of space and a harmonic function of time

$$u_x = \overline{u_x}(x, y, z)e^{i\omega t}$$

Equation 3-4a

$$u_y = \overline{u_y}(x, y, z)e^{i\omega t}$$

Equation 3-4b

$$u_z = \overline{u_z}(x, y, z)e^{i\omega t}$$

Equation 3-4c

The use of the notation

$$\overline{u} = \left\{ \begin{array}{c} \overline{u_x} \\ \overline{u_y} \\ \overline{u_z} \end{array} \right\}$$

Equation 3-5

and $\overline{\varepsilon}$ for the space portion of the strain matrix, yields the variational equation as

$$\delta \left\{ \frac{1}{2} \iiint \{ (\overline{\varepsilon}^T C \overline{\varepsilon} - \omega^2 \, \rho \overline{u}^T \, \overline{u}) dv \} \right\} = 0$$

Equation 3-6

The usual finite element approximations then lead to a set of homogeneous simultaneous equations of the form

$$[K - \omega^2 \, M]\{q\} = \{0\}$$

Equation 3-7

where K is the static stiffness matrix of the structure and M is a mass matrix. If the interpolation functions for displacements are used to determine the mass matrix M, the result will be banded. It is called the consistent mass matrix since it is consistent with the assumptions used to determine K. Sometimes a diagonal matrix M, called a lumped mass matrix, is used. This implies that the mass of the structure is concentrated at nodal points. There are no terms in the mass matrix for those equations corresponding to minimization with respect to nodal displacement derivatives so that nodal displacement derivatives may be expressed in terms of nodal displacements and eliminated from the equations. Diagonalization of the mass matrix may also

be obtained by using a diagonal value which is the sum of all of the mass matrix elements in a given row and avoids the elimination of nodal displacement derivatives.

For values of q other than zero to exist, the matrix of coefficients of Eqs. (3.7) must be singular. Thus the characteristic equation for the vibration frequencies ω is given by

$$\det\left|K - \omega^2\, M\right| = 0$$

Equation 3-8

The corresponding nodal values q which determine the vibration mode shapes are obtained by eliminating one of Eqs. (3.7) and solving for N-1 of the elements of q in terms of the Nth element. The vibration modes may be normalized to satisfy the weighted orthogonality condition

$$\left\{q_i\right\}^{T}[M]\left\{q_i\right\} = \delta_{ij}$$

Equation 3-9a

where

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

Equation 3-9b

3.3. Solution of Linear Eigenvalue Problems

The set of equations described by

$$\left[K - \omega^2\, M\right]\{q\} = 0$$

Equation 3-10

is called a linear eigenvalue problem. The characteristic equation for the eigenvalue ω^2 is given by

$$\det\left|K - \omega^2\, M\right| = 0$$

Equation 3-11

which, if expanded, would yield a polynomial equation for ω^2. The order of the polynomial is equal to the order of the matrices involved. Unless the matrices are of low order, expansion is not attempted. In any case, exact solutions are known only for polynomials of fourth order or less. Thus iterative numerical methods must be used. There are numerous methods [11], some of which are discussed below.

3.3.1. Subspace Iteration[11, 13]

When the number of eigenvalues of the system of equations is large, determination of all them may be very time consuming. In many cases, a smaller number of eigenvalues and eigenvectors may be sufficient and alternative methods are used. One of these is the method of subspace iteration in which an initial set of mode shapes which are likely to represent the important modes of the structure is chosen. If the eigenvalue problem is of nth order the number of modes chosen is m<<n. These modes are arranged by column to produce the matrix <1>1 which is of order n x m. The equation

$$[K]_{nxn} \quad [T_2]_{nxm} = [M]_{nxn} \quad [\phi_1]_{nxm}$$

Equation 3-12

is then solved for T_2. Define now

$$K^{(2)}_{(mxm)} = T_2^T K T_2$$

Equation 3-13a

$$M^{(2)}_{(mxm)} = T_2^T M T_2$$

Equation 3-14a

and solve, say by the Jacobi method, for the eigenvalues and eigenvectors of the system

$$K^{(2)} V_2 = M^{(2)} V_2 \Lambda_2$$

Equation 3-15

An improved set of eigenvectors is now

$$\phi_2 = T_2^T V_2$$

Equation 3-16

The process is repeated using the operations

$$K T_{k+1} = M \phi_k$$

Equation 3-17a

$$K^{(k+1)} = T_{k+1}^T K T_{k+1}$$

Equation 3-17b

$$M^{(k+1)} = T_{k+1}^T M T_{k+1}$$

Equation 3-17c

$$K^{(k+1)}V_{k+1} = M^{(k+1)}V_{k+1}\Lambda_{k+1}$$

Equation 3-17d

$$\phi_{k+1} = T^T_{k+1}V_{k+1}$$

Equation 3-17e

until convergence is achieved.

A check on whether the desired eigenvalues and eigenvectors have been obtained is afforded by the S turm sequence property of the eigenvalue problem which states that if the matrix $[K - \omega^2 M]$ with a given value of ω^2 is decomposed into the form

$$\left[K - \omega^2 M\right] = S^T S$$

Equation 3-18

where S is an upper triangular matrix, the number of eigenvalues less than ω^2 is equal to the number of negative diagonal elements of S . Thus if the subspace iteration method has produced m eigenvectors which are supposedly the m lowest eigenvectors, then the use of a value of ω^2 slightly greater than ω^2_m should produce exactly m negative diagonal values of S.

3.3.2. Lanczos Algorithm

A method of eigenvalue extraction which has been found to involve considerably less computation time for large eigenvalue problems is the Lanczos algorithm. The stiffness matrix K is decomposed into the form

$$K = LL^T$$

Equation 3-19

where L is a lower triangular matrix. The substitutions

$$Y = L^T q$$

Equation 3-20a

$$\bar{\lambda} = \frac{1}{\omega^2}$$

Equation 3-20b

transform the eigenvalue problem to the form

$$AY = \bar{\lambda}Y$$

Equation 3-21a

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with

$$A = \left(L^{-1}\right)M\left(L^{-1}\right)^{T}$$

Equation 3-21b

The matrix A is then transformed to a symmetric tridiagonal form with the use of an orthogonal matrix V such that

$$V^{T}AV = T$$

Equation 3-22

with T tridiagonal. The substitution

$$Y = VQ$$

Equation 3-23

results in

$$TQ = \begin{bmatrix} \alpha_1 & \beta_1 & & & & & & \\ \beta_1 & \alpha_2 & \beta_2 & & & & & \\ & \beta_2 & \alpha_3 & \beta_3 & & & & \\ & & \beta_3 & \alpha_4 & . & & & \\ & & & . & . & . & & \\ & & & & . & . & . & \\ & & & & & \beta_{n-2} & \alpha_n & \beta_{n-1} \\ & & & & & & \beta_{n-1} & \alpha_n \end{bmatrix} \qquad Q = \frac{1}{\lambda}Q$$

Equation 3-24

which can be solved easily for accurate eigenvalues using a determinant search method (the bisection method) in conjunction with the Sturm sequence property of the eigenvalues. Not all of the n eigenvalues are found. The number of eigenvector components and equations is usually truncated to a value m<n, where m is twice the number of desired eigenvalues. The corresponding eigenvectors Q may be found by the method of inverse iteration.

The column vectors of the matrix V are generated so that each is orthogonal to the two preceding vectors. From the definition

$$AV = VT$$

Equation 3-25a

can be obtained the relation

$$AV_i = \beta_{i-1}V_{i-1} + \alpha_i\ V_i + \beta_i\ V_{i+1}$$

Equation 3-25b

Where V_j is the jth column vector of V and satisfies the relation

$$V_i^T V_j = \delta_{ij}$$

Equation 3-25c

If V_i and V_{i-1} are known, then

$$\alpha_i = V_i^T A V_i$$

Equation 3-26a

$$\beta_i = \left(W_i^T W_i\ \right)^{1/2}$$

Equation 3-26b

with

$$W_i = AV_i - \beta_{i-1}V_{i-1} - \alpha_i\ V_i$$

Equation 3-26c

and

$$V_{i+1} = \frac{1}{\beta_i} W_i$$

Equation 3-26d

The sequence of operations is started by using the first column of the unit vector as V_1 and β_0 taken as zero.

After a number of steps, the Lanczos vectors V_i may lose orthogonality because of truncation of the number of eigenvalues and will require reorthogonalization. Vector reorthogonalization is achieved by a procedure which can be summarized as follows

$$S_{i+1} = P_i\ P_{i-1}\ ...\ P_1\ W_i$$

Equation 3-27a

$$P_{i+1} S_{i+1} = \beta_i\ e_{i+1}$$

Equation 3-27b

$$V_{i+1} = P_1 \; P_2 \quad \dots \; P_{i+1} e_{i+1}$$

Equation 3-27c

with

$$P_1 \;\; = I$$

Equation 3-27d

and e_j the jth column of the unit matrix. The eigenvectors of the original equation are then

$$q = \left(L^{-1}\right)^T P_1 \; P_2 \; \dots . P_m \; \frac{Q}{0}$$

Equation 3-28

Chapter 4. Buckling of Structures

4.1.   The Phenomenon of Buckling

In the linear theory of structural analysis, the behavior of a structure under a given loading is unique. For specified loading and support conditions the structure can deform in only one way and have only one internal stress state. For sufficiently large loads the nonlinear aspects of structural behavior can no longer be ignored. One of the causes of nonlinearity is nonlinear material behavior for which Hooke's law no longer applies. It is possible, however, for the structure to behave in a nonlinear fashion while the material is still in the elastic range. This is especially true for structures for which one dimension is small compared to the others, such as in long beams or thin plates and shells.

One of the phenomena which may occur is that of buckling. The classic example is that of an axially compressed initially straight beam which is found to have two distinct equilibrium positions, the straight position and a deflected position, when the load exceeds a certain critical value. Similarly, an initially flat plate under inplane loading can deflect laterally and remain in equilibrium when the load exceeds a critical value. In both of these cases the critical load is a reasonably accurate measure of the load below which deflections will not become excessive. Thin shells are also subject to buckling, but the effect of small initial deviations from the idealized shape can result in actual critical loads which are very much less than those calculated theoretically. For these structures recourse is usually had to empirical "knockdown factors" by means of which the theoretical load is reduced.

4.2.   Calculation of Critical Loads

Critical loads are calculated by considering a structure which has an initial stress and deformation state due to some distribution of externally applied loads with a magnitude governed by a proportionality factor λ. When the linear theory of elasticity is used, the calculated initial stress and deformation states are proportional to the external loading and thus have a magnitude which varies linearly with λ. The equilibrium of the structure when arbitrary infinitesimal disturbances of the initial deflection state are superimposed is then investigated. The equations of equilibrium are linearized with respect to the small disturbances so that their solution by means of the finite element method leads to a set of simultaneous linear equations for the modal unknowns of the form

$$[K_B - \lambda K_G]\{q\} = \{0\}$$

Equation 4-1

where K_B is the usual structural stiffness matrix in the absence of applied loading while K_G is a "geometric stiffness matrix" which is independent of the material properties of the structure.

For solutions other than the initial stress state to exist, i.e.,

$$\{q\} = \{0\}$$

Equation 4-2

the determinant of the coefficients matrix must vanish. Then an equation for λ is given by

$$\det[K - \lambda K_G] = 0$$

Equation 4-3

The lowest value of λ which satisfies equation (4.3) is called the critical value, the value at which the structure can suddenly undergo large deformations which differ from the expected deformation state under the system of loading. The corresponding distribution of nodal values of q is called the buckling mode shape. Relative values of q may be calculated by deleting one equation from Eqs (4.1) and solving for the ratio of N- 1 of the nodal values and the Nth nodal value.

4.3. Variational Principles for Buckling

A variational principle from which critical loads can be obtained is given by

$$\delta \pi_B = 0$$

Equation 4-4

with

$$\pi_B = \frac{1}{2} \iiint\limits_V \left(\varepsilon^T C \varepsilon + \lambda \Delta^T \overline{\sigma} \Delta \right) dv$$

Equation 4-5a

and

$$\Delta^T = \left\{ \frac{\partial u_x}{\partial x} \quad \frac{\partial u_x}{\partial y} \quad \frac{\partial u_x}{\partial z} \quad \frac{\partial u_y}{\partial x} \quad \frac{\partial u_y}{\partial y} \quad \frac{\partial u_y}{\partial z} \quad \frac{\partial u_z}{\partial x} \quad \frac{\partial u_z}{\partial y} \quad \frac{\partial u_z}{\partial z} \right\}$$

Equation 4-6b

$$\overline{\sigma} = \begin{bmatrix} \sigma_{xx}^0 & \sigma_{xy}^0 & \sigma_{xz}^0 & & & & & & \\ & \sigma_{yy}^0 & \sigma_{yz}^0 & & & & & & \\ & & \sigma_{xx}^0 & & & & & & \\ & & & \sigma_{xx}^0 & \sigma_{xy}^0 & \sigma_{xz}^0 & & & \\ & & & & \sigma_{yy}^0 & \sigma_{yz}^0 & & & \\ & & & & & \sigma_{zz}^0 & & & \\ & & & & & & \sigma_{xx}^0 & \sigma_{xy}^0 & \sigma_{xz}^0 \\ & & & & & & & \sigma_{yy}^0 & \sigma_{yx}^0 \\ \text{sym} & & & & & & & & \sigma_{zz}^0 \end{bmatrix}$$

Equation 4-5c

The superscript ^{0} on the stresses denotes a distribution calculated from linear elasticity theory. The coefficient A is the proportionality factor by which the calculated linear stress state must be multiplied for buckling to occur. The initial stress state may be the superposition of stress distributions with different proportionality factors, i.e.,

$$\overline{\sigma} = \sum_{i=1}^{P} \lambda_i \overline{\sigma}_i$$

Equation 4-6

in which case the variational principle becomes

$$\delta \frac{1}{2} \iiint_v \left(\varepsilon^T C \varepsilon + \sum_{i=1}^{P} \lambda_i \Delta^T \overline{\sigma}_i \Delta \right) dv = 0$$

Equation 4-7

The variational principle may be specialized for various types of structures and load conditions.

4.3.1. Inplane Buckling for Plane Stress, Plane Strain, Axisymmetric Stress States

For a body in a state of plane stress or plane strain which is subject only to inplane buckling,

$$u_z = \sigma_{xz}^0 = \sigma_{yz}^0 = 0$$

while u_x and u_y are functions of x and y only. Then for a orthotropic material

$$\varepsilon^T = \left\{ \frac{\partial u_x}{\partial x} \frac{\partial u_y}{\partial y} \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right\}$$

Equation 4-8a

$$C = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} \\ & C'_{22} & C'_{23} \\ \text{sym} & & C'_{33} \end{bmatrix}$$

Equation 4-8b

$$\Delta^T = \left\{ \frac{\partial u_x}{\partial x} \frac{\partial u_x}{\partial y} \frac{\partial u_y}{\partial x} \frac{\partial u_y}{\partial y} \right\}$$

Equation 4-8c

$$\overline{\sigma} = \begin{bmatrix} \sigma^0_{xx} & \sigma^0_{xy} & 0 & 0 \\ & \sigma^0_{yy} & 0 & 0 \\ & & \sigma^0_{xx} & \sigma^0_{xy} \\ \text{sym} & & & \sigma^0_{yy} \end{bmatrix}$$

Equation 4-8d

$$\pi_B = \frac{1}{2} \iint_a \left(\varepsilon^T C \varepsilon + \lambda \Delta^T \overline{\sigma} \Delta \right) da$$

Equation 4-9

where the integration is over the area of the body.

For an axisymmetric body with an initial axisymmetric stress state and which buckles axisymmetrically, the function 1tB is of the same form as given in Eq. (4.9). However, the matrices in that expression are now defined by

$$\varepsilon^T = \left\{ \frac{\partial u_r}{\partial r} \frac{\partial u_z}{\partial z} \frac{u_r}{r} \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right\}$$

Equation 4-10a

$$C = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & C'_{14} \\ & C'_{22} & C'_{23} & C'_{24} \\ & & C'_{33} & C'_{34} \\ \text{sym} & & & C'_{44} \end{bmatrix}$$

Equation 4-10b

$$\Delta^T = \left\{ \frac{\partial u_r}{\partial r} \frac{\partial u_r}{\partial z} \frac{\partial u_z}{\partial r} \frac{\partial u_z}{\partial z} \frac{u_r}{r} \right\}$$

Equation 4-10c

$$\overline{\sigma} = \begin{bmatrix} \sigma_{rr}^0 & \sigma_{rz}^0 & & & \\ & \sigma_{zz}^0 & & & \\ & & \sigma_{rr}^0 & \sigma_{rz}^0 & \\ & & & \sigma_{zz}^0 & \\ \text{sym} & & & & \sigma_{\phi\phi}^0 \end{bmatrix}$$

Equation 4-10d

4.3.2. Straight Beams

For buckling of straight beams with shear deformations included, the usual assumptions of beam theory are made. Then, with x, y, and z denoting the centroidal longitudinal axis and the centroidal principle axes, respectively,

$$\pi_B = U - \frac{1}{2} \int_0^L P \left[\left(\frac{\partial u_x}{\partial x} \right)^2 + \left(\frac{\partial u_y}{\partial x} \right)^2 + \left(\frac{\partial u_z}{\partial x} \right)^2 \right] dx$$

Equation 4-11

where the strain energy U is given by Eqs. (2.1 2) and (2.14) and P is the axial load calculated from a linear analysis and assumed positive in compression. With this formulation, the beam element may have an arbitrary orientation in space when the proper axis rotations are made. The form of the variational equation implies that interpolation functions chosen for displacements and rotations need only satisfy C^0 continuity at element edges.

For overall buckling of trusses, the members are assumed to change orientation but to remain straight. Then π_B may be written as

$$\pi_B = \frac{1}{2} \int_0^L \left\{ EA \left(\frac{\partial u_x}{\partial x} \right)^2 - P \left[\left(\frac{\partial u_x}{\partial x} \right)^2 + \left(\frac{\partial u_y}{\partial x} \right)^2 + \left(\frac{\partial u_z}{\partial x} \right)^2 \right] \right\} dx$$

Equation 4-12

In all functionals, the term $P \left(\frac{\partial u_x}{\partial x} \right)^2$ may be deleted since

$$\frac{P}{EA} = \frac{\sigma_0}{E} \ll 1$$

Equation 4-13

4.3.3. Flat Plates

Assumptions similar to those for linear analysis of flat plates are made for buckling of flat plates. Then the functional π_B may be written for an isotropic plate as

$$
\pi_B = U + \frac{1}{2} \int \left\{ \frac{\partial u_z}{\partial x} \; \frac{\partial u_z}{\partial y} \right\} \begin{bmatrix} \overline{N}_{xo} & \overline{N}_{xyo} \\ \overline{N}_{xyo} & \overline{N}_{yo} \end{bmatrix} \left\{ \begin{array}{c} \dfrac{\partial u_z}{\partial x} \\ \dfrac{\partial u_z}{\partial y} \end{array} \right\} dxdy
$$

Equation 4-14

where U is the strain energy given by Eqs. (2.19) to (2.22) and $\overline{N}_{xo}$, $\overline{N}_{yo}$, $\overline{N}_{xyo}$ are inplane stress resultants in the plate prior to buckling. If shearing deformations are neglected for thin plates, the strain energy function is replaced by Eqs. (2.19), (2.25), and (2.26). In the former formulation C^0 continuity is required whereas in the latter C^1 continuity is needed.

4.4. Calculation of Eigenvalues

The lowest buckling load is generally the only one of interest so that use of the inverse iteration method is indicated. Any of the other eigenvalue extraction methods can be used, however.

Chapter 5. Heat Transfer

5.1. Equations of Heat Transfer[13]

Let q_x, q_y, and q_z be the heat flux in a body in the x, y, and z directions and Q the internally generated heat flow. Then the heat balance equation is given by

$$Q - \frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} = c\rho \frac{\partial T}{\partial t}$$

Equation 5-1

where T is temperature, c is specific heat, and ρ is mass density.

The Fourier heat conduction equation relates heat flux and temperature for a thermally orthotropic body with axes of orthotropy coinciding with the coordinate axes as

$$q_{x'} = -k_{x'} \frac{\partial T}{\partial x'}$$

Equation 5-2a

$$q_{y'} = -k_{y'} \frac{\partial T}{\partial y'}$$

Equation 5-2b

$$q_{z'} = -k_{z'} \frac{\partial T}{\partial z'}$$

Equation 5-2c

with $k_{x'}$, $k_{y'}$, and $k_{z'}$ the thermal conductivities in the principal thermal orthotrophy directions. These equations can be expressed in matrix form as

$$\begin{Bmatrix} q_{x'} \\ q_{y'} \\ q_{z'} \end{Bmatrix} = -[k'] \begin{Bmatrix} \dfrac{\partial T}{\partial x'} \\ \dfrac{\partial T}{\partial y'} \\ \dfrac{\partial T}{\partial z'} \end{Bmatrix}$$

Equation 5-3

with

$$[k'] = \begin{bmatrix} k_{x\prime} & 0 & 0 \\ 0 & k_{y\prime} & 0 \\ 0 & 0 & k_{z\prime} \end{bmatrix}$$

Equation 5-4

The three components of heat conduction can be considered to be components of the heat flux vector q. Similarly the three derivatives of T are the three components of the temperature gradient vector. Under a rotation of the coordinate system, then, the Fourier heat conduction relations become

$$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = -[k] \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \\ \dfrac{\partial T}{\partial z} \end{Bmatrix}$$

Equation 5-5

where

$$[k] = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ & k_{yy} & k_{yz} \\ \text{sym} & & k_{zz} \end{bmatrix} = N^T[k']N$$

Equation 5-6

with N the rotation matrix given by Eq. (1.8). When the coordinate axis rotation consists of a clockwise rotation through an angle θ about the z-axis, the rotation matrix becomes

$$N = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equation 5-7

and

$$[k] = \begin{bmatrix} k_{x\prime}\cos^2\theta + k_{y\prime}\sin^2\theta & (k_{x\prime} - k_{y\prime})\sin\theta\cos\theta & 0 \\ & k_{x\prime}\sin^2\theta + k_{y\prime}\cos^2\theta & 0 \\ \text{sym} & & k_{z\prime} \end{bmatrix}$$

Equation 5-8

The heat balance equation may now be written as

$$\left\{\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z}\right\}[k]\left\{\begin{array}{c}\dfrac{\partial T}{\partial x}\\[6pt]\dfrac{\partial T}{\partial y}\\[6pt]\dfrac{\partial T}{\partial z}\end{array}\right\} + Q = c\rho\frac{\partial T}{\partial t}$$

Equation 5-9

When cylindrical coordinates r, ϕ, and z are used, as for a body of revolution, the heat balance equation is readily derived as

$$\left\{\frac{\partial}{\partial r} \frac{1}{r}\frac{\partial}{\partial \phi} \frac{\partial}{\partial z}\right\}[k]\left\{\begin{array}{c}\dfrac{\partial T}{\partial r}\\[6pt]\dfrac{1}{r}\dfrac{\partial T}{\partial \phi}\\[6pt]\dfrac{\partial T}{\partial z}\end{array}\right\} + Q = c\rho\frac{\partial T}{\partial t}$$

Equation 5-10

The material is usually assumed to be thermally orthotropic with one axis of orthotropic coinciding with the circumferential ϕ direction. Then [k] is of the form of Eq. (5.8) with $k_{x'}$ and $k_{y'}$ replaced by the equivalent conductivities in the r-z plane while $k_{z'}$ is replaced by the conductivity k_{ϕ}.

Boundary conditions that may be imposed are the following:

(a) temperature T^* may be prescribed on portion S_T of the boundary S

(b) heat flux q^* may be prescribed on portion S_T of the boundary S. Since heat flow occurs normal to the boundary, the prescribed heat flux is given by

$$\vec{q}.\vec{n} = q_x n_x + q_y n_y + q_z n_z = -\{n_x n_y n_z\}[k]\left\{\begin{array}{c}\dfrac{\partial T}{\partial x}\\[6pt]\dfrac{\partial T}{\partial y}\\[6pt]\dfrac{\partial T}{\partial z}\end{array}\right\} = q^*$$

Equation 5-11a

with $\vec{n}$ the vector normal to the surface S_q and n_x, n_y, and n_z its components. If the surface is insulated

$$q^* = 0$$

Equation 5-11b

(c) convective heat transfer conditions may be prescribed on portion S_C of S. Then

$$\vec{q}.\vec{n} = h(T - T_\infty)$$

Equation 5-12

with h the convective heat transfer or film coefficient and T_∞ the fluid temperature.

In addition, the temperature distribution within the body must be prescribed at time $t=0$.

5.2.   Variational Statement and the Finite element Method

An equivalent variational principle valid at every instant of time may be written as

$$\delta \pi_T = 0$$

Equation 5-13

where

$$\pi_r = \iiint_V \left(\frac{1}{2} \begin{Bmatrix} \partial T/\partial x \\ \partial T/\partial y \\ \partial T/\partial z \end{Bmatrix}^T [k] \begin{Bmatrix} \partial T/\partial x \\ \partial T/\partial y \\ \partial T/\partial z \end{Bmatrix} - QT - \rho c T \frac{\partial T}{\partial t} \right) dxdydz + \int_{S_q} q^* T dS$$
$$+ \int_{S_c} h \left(\frac{1}{2} T^2 - T T_\infty \right) ds$$

Equation 5-14

and dS is an element of area on the surface S. In this formulation $\partial T/\partial t$ is not subject to variation.

The body is now divided into elemental subregions and the temperature field within the element is represented by

$$T = \{D\}\{T_n\}$$

Equation 5-15

where [D] is a row vector of interpolation or shape functions which depend on the position in the element and $\{T_n\}$ is a column vector of nodal temperatures and possible derivatives of the temperatures. The shape function need only satisfy C^0continuity, i.e., only the function itself need be continuous at element boundaries.

Then

$$\left\{\begin{array}{c} \dfrac{\partial T}{\partial x} \\[2mm] \dfrac{\partial T}{\partial y} \\[2mm] \dfrac{\partial T}{\partial z} \end{array}\right\} = [B]\{T_n\}$$

Equation 5-16

with

$$[B] = \left\{\begin{array}{c} \dfrac{\partial D}{\partial x} \\[2mm] \dfrac{\partial D}{\partial y} \\[2mm] \dfrac{\partial D}{\partial z} \end{array}\right\}$$

Equation 5-17

The portion of π_T contributed by each element is then given by

$$\Delta\pi_T = \{T_n\}^T \left\{\frac{1}{2}([k_T] + [h_T])\{T_n\} - \{r_Q\} + \{r_q\} - [C_T]\left\{\frac{\partial T_n}{\partial t}\right\} - \{r_\infty\}\right\}$$

Equation 5-18

with

$$[k_T] = \iiint\limits_{\Delta V} [B]^T[k][B]dV$$

Equation 5-19a

$$[h_T] = \iint\limits_{\Delta S_c} h\,[D]^T[D]dS$$

Equation 5-19b

$$\{r_Q\} = \iiint\limits_{\Delta V} Q[D]^T \, dV$$

Equation 5-19c

$$\{r_q\} = \iint\limits_{\Delta S_q} q^* \, [D]^T \, dS$$

Equation 5-19d

$$[C_T] = \iiint\limits_{\Delta V} \rho c [D]^T \, [D] dV$$

Equation 5-19e

$$\{r_\infty\} = \iint\limits_{\Delta S_c} h T_\infty \, [D]^T \, dS$$

Equation 5-19f

Here ΔV is the volume of the element, ΔS_c is that portion of the boundary S_c which is a boundary of the element, and ΔS_q is that portion of the boundary S_q which is a boundary of the element. For interior elements these are equal to zero. On the remaining portions of the boundary S_T, the temperature at nodal points is given by T^*

Assembly of the variational functional Π_T yields an expression of the form

$$\pi_T = \{T\}^T \left\{ \frac{1}{2}\{[K_T] + [H_T]\}\{T\} - \{R_Q\} + \{R_q\} + [C_T]\left\{\frac{\partial T}{\partial t}\right\} - \{R_\infty\}\right\}$$

Equation 5-20

Where $\{T\}$ is the matrix of nodal temperatures arranged sequentially and the remaining matrices are the assembled versions of those defined by Eqs. (5.19). Minimization with respect to the nodal quantities then yields the set of equations

$$[C_T]\left[\frac{\partial T}{\partial t}\right] + [K]\{T\} = \{R\}$$

Equation 5-21

with

$$[K] = \left[K_T\right] + \left[H_T\right] \qquad \{R\} = \{R_Q\} + \{R_\infty\} - \{R_q\}$$

Equation 5-22

Those equations of the above set which represent minimization with respect to nodal temperatures on S_T should be deleted. In the remaining equations those terms of $\partial T/\partial t$ referring to temperatures on S_T vanish while the corresponding terms in T are set equal to the prescribed temperature. This approach can be applied easily if the prescribed temperature is zero since the node at which that temperature is prescribed need not be included in the numbering system. For a non-zero prescribed nodal temperature T^*, another approach is to add a large value of conductivity K_n to the corresponding diagonal coefficient of $[K]$ and to replace the corresponding coefficient in $\{R\}$ by $K_n T_n^*$ · This method effectively forces the nodal temperature to be equal to the prescribed value for sufficiently large K_n.

5.3.   Solution of Transient Heat Conduction[13]

The analysis of steady-state heat conduction problems involves the solution of a set of simultaneous equations given by Eqs. (5.21) with each term of $\partial T/\partial t$ set equal to zero and with Q and q^* assumed to be independent of time. This may be accomplished, as for static stress · analysis, by Gaussian elimination. The analysis of transient heat conduction, however, involves the solution of Eqs. (5.21) as a set of simultaneous first order linear differential equations in time subject to certain initial conditions.

These equations may be solved by mode superposition, as for dynamic stress problems, by determining the eigenvalues and eigenvectors of the equation

$$[K - \lambda C_T]\{T\} = \{0\}$$

Equation 5-23

A more usual approach is to numerically integrate the differential equations. One such method uses the assumption that temperatures at time t and t+Δt are related by

$$\{T\}_{t+\Delta t} = \{T\}_t + \left\{ (1-\beta)\left\{\frac{\partial T}{\partial t}\right\}_t + \beta\left\{\frac{\partial T}{\partial t}\right\}_{t+\Delta t} \right\}$$

Equation 5-24

By writing Eqs. (5.21) for time t and t+Δt, the derivatives of temperature can be eliminated and a set of equations for temperature at time t+Δt can be obtained as

$$\left(\frac{1}{\Delta t}[C_T] + \beta[K]\right)\{T\}_{t+\Delta t} = \left(\frac{1}{\Delta t}[C_T] + (1-\beta)[K]\right)(T)_t + (1-\beta)\{R\}_t + \beta\{R\}_{t+\Delta t}$$

Equation 5-25

Thus the problem is reduced to the repeated solution of a set of simultaneous equations. For constant .0.t the matrix on the left side of the equation is independent of time and need be reduced by Gaussian elimination only once. With given initial values of {T} at t=0 and with $\{R\}_{t=0}$ at $\{R\}_{t=\Delta t}$ known, the set of equations may be solved for $\{T\}_{t=\Delta t}$. The right side is then changed using the new values of $\{T\}_{t=\Delta t}$ and the values of $\{R\}_{t=\Delta t}$ and $\{R\}_{t=2\Delta t}$ and the solution obtained for $\{T\}_{t=2\Delta t}$ and so on.

In the program β is taken as 1, in which case the method is known as the Euler method. The method is unconditionally stable if Δt is less than $2/\Delta_{max}$. where λ_{max} is the largest eigenvalue of Eq. (5.23).

Chapter 6. The Element Library

A number of elements for different uses are available in the program. These are discussed below.
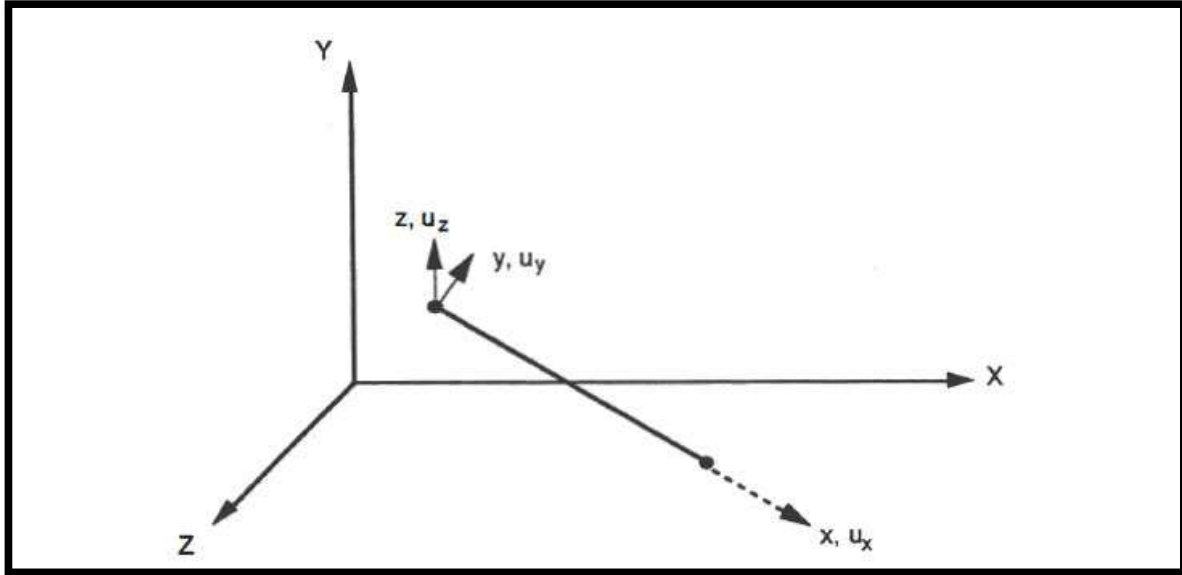
6.1. TRUSS3D: Linear 3-D Truss/Spar

FIGURE 15

Matrix	Variational Function	Shape Function	Integration
Geometric Stiffness $[k'_G]$	$\dfrac{1}{2}\displaystyle\int_0^{\Delta L} P_x\left[\left(\dfrac{\partial u_x}{\partial x}\right)^2 + \left(\dfrac{\partial u_Y}{\partial x}\right)^2 + \left(\dfrac{\partial u_Z}{\partial x}\right)^2\right]dx$ P_x constant	$u_X = u_{X1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{X2}\left(\dfrac{x}{\Delta L}\right)$ $u_Y = u_{Y1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{Y2}\left(\dfrac{x}{\Delta L}\right)$ $u_Z = u_{Z1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{Z2}\left(\dfrac{x}{\Delta L}\right)$	Exact, no integration points
Mass $[m']$	$\dfrac{1}{2}\displaystyle\int_0^{\Delta L} \rho A\left(u_x^2 + u_y^2 + u_z^2\right)dx$ ρ, A constant	$u_X = u_{X1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{X2}\left(\dfrac{x}{\Delta L}\right)$ $u_Y = u_{Y1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{Y2}\left(\dfrac{x}{\Delta L}\right)$ $u_Z = u_{Z1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{Z2}\left(\dfrac{x}{\Delta L}\right)$	Exact, no integration points
Gravitational Loading P_g	$\displaystyle\int_0^{\Delta L} \rho A\left(g_x u_x + g_y u_y + g_z u_z\right)dx$ ρ, A, g_X, g_Y, g_Z constant	$u_X = u_{X1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{X2}\left(\dfrac{x}{\Delta L}\right)$ $u_Y = u_{Y1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{Y2}\left(\dfrac{x}{\Delta L}\right)$ $u_Z = u_{Z1}\left(1 - \dfrac{x}{\Delta L}\right) + u_{Z2}\left(\dfrac{x}{\Delta L}\right)$	Exact, no integration points

Transformed Linear Stiffness Matrix:

$$k_B = T^T k'_B T$$

with

$$T = \begin{bmatrix} \dfrac{\Delta X}{\Delta L} & \dfrac{\Delta Y}{\Delta L} & \dfrac{\Delta Z}{\Delta L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\Delta X}{\Delta L} & \dfrac{\Delta Y}{\Delta L} & \dfrac{\Delta Z}{\Delta L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\Delta X = X_2 - X_1$$

$$\Delta Y = Y_2 - Y_1$$

$$\Delta Z = Z_2 - Z_1$$

$$\Delta L = \sqrt{\Delta X^2 + \Delta Y^2 + \Delta Z^2}$$

Stress:

$$\sigma_x = E\left(\frac{u_{x2} - u_{x1}}{\Delta L} - \alpha \Delta T\right)$$

$$\sigma_x = E\left(\begin{bmatrix} -\dfrac{\Delta X}{\Delta L^2} & -\dfrac{\Delta Y}{\Delta L^2} & -\dfrac{\Delta Z}{\Delta L^2} & \dfrac{\Delta X}{\Delta L^2} & \dfrac{\Delta Y}{\Delta L^2} & \dfrac{Z}{\Delta L^2} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{z1} \\ u_{x2} \\ u_{y2} \\ u_{z2} \end{Bmatrix} - \alpha \Delta T \right)$$

Transformed Thermal Loading:

$$P_T = \left\{ \begin{array}{c} \dfrac{\Delta X}{\Delta L} \\[2mm] \dfrac{\Delta Y}{\Delta L} \\[2mm] \dfrac{\Delta Z}{\Delta L} \\[2mm] -\dfrac{\Delta X}{\Delta L} \\[2mm] -\dfrac{\Delta Y}{\Delta L} \\[2mm] -\dfrac{\Delta Z}{\Delta L} \end{array} \right\} p'_T$$
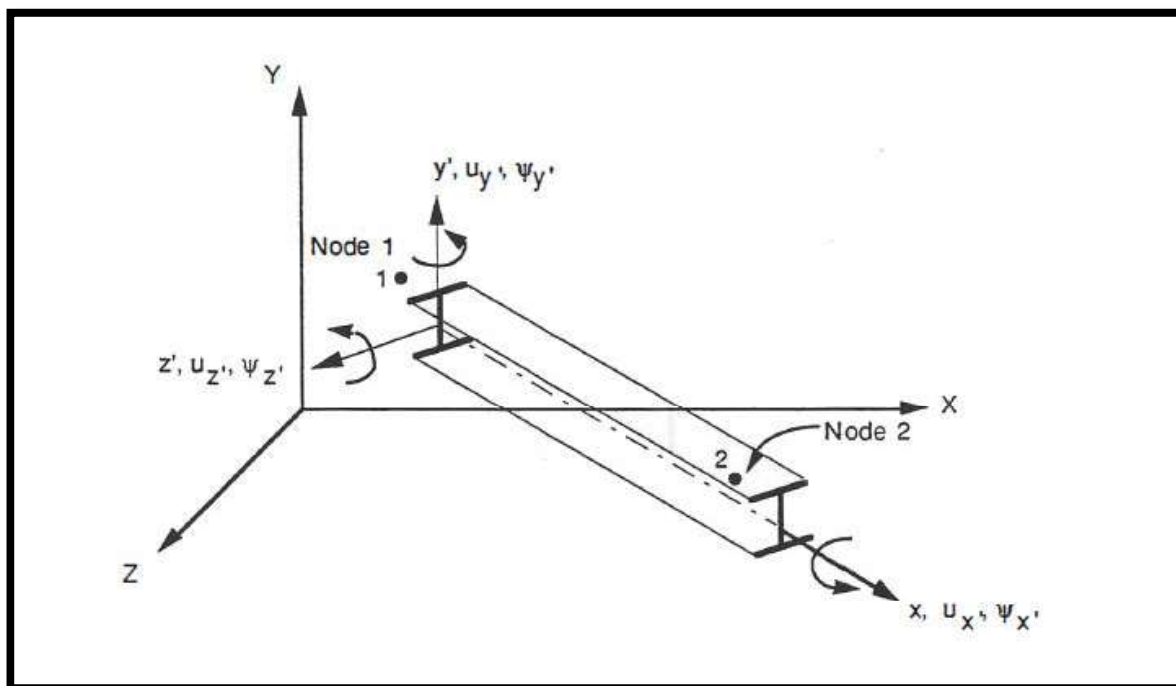
6.2. BEAM3D:Linear 3-D Elastic Beam

FIGURE 16

For symmetric beams the x axis coincides with the beam centroidal axis and the y and z axes are principal centroidal axes. Node 3 lies in the principal x-y plane. For unsymmetric beams with offset, the orientation of the element x, y, and z axes is determined by the location of two offset nodes and a third node. The location of the element end nodal points with respect to the member end centroids is given in terms of member centroidal coordinate axes x', y', and z' defined such that the x' axis coincides with the member centroidal axis, the z' axis lies in the element x-z plane and is perpendicular to the x' axis, while the y' axis is perpendicular to both the x' and z' axes, the three forming a right-handed Cartesian coordinate system. Shear factors, however, are those for the member principal axes. Stiffness matrices are first calculated for member principal axes and are then transformed to account for the orientation of the various coordinate axes with respect to principal axes and for the offsets of the nodes from the centroids of the element end cross-sections. If $I_{y'y'}$, $I_{z'z'}$, and $I_{y'z'}$ are, respectively, the moments of inertia of the cross-section about the y' and z' centroidal axes and the centroidal product of inertia, the principal moments of inertia are given by

$$\left\{ \begin{matrix} \bar{I}_y \\ \bar{I}_z \end{matrix} \right\} = \begin{bmatrix} \frac{1}{2}(1 + \cos 2\theta) & \frac{1}{2}(1 - \cos 2\theta) & -\sin 2\theta \\ \frac{1}{2}(1 - \cos 2\theta) & \frac{1}{2}(1 + \cos 2\theta) & \sin 2\theta \end{bmatrix} \left\{ \begin{matrix} I_{y'y'} \\ I_{z'z'} \\ I_{y'z'} \end{matrix} \right\}$$

where

$$-90° \leq 2\theta = \tan^{-1}\frac{2I_{y'z'}}{I_{z'z'} - I_{y'y'}} \leq 90°$$

Matrix	Variational Function	Shape Functions	Integration
Linear Stiffness $[k_B]$	Eq. (2.14) for principal axes $E, G, v, A, \bar{I}_y, \bar{I}_z, \bar{J}, \bar{k}_y, \bar{k}_z$	$u_x = u_{x1}\left(1 - \frac{x}{\Delta L}\right) + u_{x2}\left(\frac{x}{\Delta L}\right)$ $u_y = \frac{1}{1 + \phi_y}\left\{\left[1 + \phi_y + \frac{x}{\Delta L} - 2\left(\frac{x}{\Delta L}\right)^2\right]\left(1 - \frac{x}{\Delta L}\right)u_{y1}\right.$ $+ \frac{x}{\Delta L}\left(1 - \frac{x}{\Delta L}\right)\left(1 + \frac{1}{2}\phi_y - \frac{x}{\Delta L}\right)\Psi_{z1}\Delta L$ $+ \left(\frac{x}{\Delta L}\right)\left[\phi_y + 3\frac{x}{\Delta L} - 2\left(\frac{x}{\Delta L}\right)^2\right]u_{y2}$ $- \frac{x}{\Delta L}\left(1 - \frac{x}{\Delta L}\right)\left(\frac{1}{2}\phi_y + \frac{x}{\Delta L}\right)\Psi_{z2}\Delta L\right\}$ $u_z = \frac{1}{1 + \phi_z}\left\{\left[1 + \phi_z + \frac{x}{\Delta L} - 2\left(\frac{x}{\Delta L}\right)^2\right]\left(1 - \frac{x}{\Delta L}\right)u_{z1}\right.$ $+ \frac{x}{\Delta L}\left(1 - \frac{x}{\Delta L}\right)\left(1 + \frac{1}{2}\phi_z - \frac{x}{\Delta L}\right)\Psi_{y1}\Delta L$ $+ \left(\frac{x}{\Delta L}\right)\left[\phi_z + 3\frac{x}{\Delta L} - 2\left(\frac{x}{\Delta L}\right)^2\right]u_{z2}$ $- \frac{x}{\Delta L}\left(1 - \frac{x}{\Delta L}\right)\left(\frac{1}{2}\phi_z + \frac{x}{\Delta L}\right)\Psi_{y2}\Delta L\right\}$ $\Psi_x = \Psi_{x1}\left(1 - \frac{x}{\Delta L}\right) + \Psi_{x2}\left(\frac{x}{\Delta L}\right)$ $\Psi_y = \frac{1}{1 + \phi_z}\left[\left(1 - \frac{x}{\Delta L}\right)\left(1 + \phi_z - 3\frac{x}{\Delta L}\right)\Psi_{y1}\right.$ $- \left(\frac{x}{\Delta L}\right)\left(2 - \phi_z - 3\frac{x}{\Delta L}\right)\Psi_{y2}$ $\left. - 6\frac{x}{\Delta L}\left(1 - \frac{x}{\Delta L}\right)\left(\frac{u_{z2} - u_{z1}}{\Delta L}\right)\right]$ $\Psi_z = \frac{1}{1 + \phi_z}\left[\left(1 - \frac{x}{\Delta L}\right)\left(1 + \phi_y - 3\frac{x}{\Delta L}\right)\Psi_{z1}\right.$ $- \left(\frac{x}{\Delta L}\right)\left(2 - \phi_y - 3\frac{x}{\Delta L}\right)\Psi_{z2}$ $\left. - 6\frac{x}{\Delta L}\left(1 - \frac{x}{\Delta L}\right)\left(\frac{u_{y2} - u_{y1}}{\Delta L}\right)\right]$	Exact, no integration points

		$\phi_y = \dfrac{24(1+v)\bar{I}_z}{\bar{k}_y A L^2}$ $\qquad$ $\phi_z = \dfrac{24(1+v)\bar{I}_y}{\bar{k}_z A L^2}$ (x, y, z principal axes)	
Geometric Stiffness [k_G]	$\dfrac{1}{2}\displaystyle\int_0^{\Delta L} P_x\left[\left(\dfrac{\partial u_y}{\partial x}\right)^2 + \left(\dfrac{\partial u_z}{\partial x}\right)^2\right]dx$ P_x constant	$u_y = \left(1 - \dfrac{x}{\Delta L}\right)^2\left[\left(1 + 2\dfrac{x}{\Delta L}\right)u_{y1} + \dfrac{x}{\Delta L}\Psi_{z1}\Delta L\right]$ $\qquad + \left(\dfrac{x}{\Delta L}\right)^2\left[\left(3 - 2\dfrac{x}{\Delta L}\right)u_{y2}\right.$ $\qquad\qquad \left. - \left(1 - \dfrac{x}{\Delta L}\right)\Psi_{z2}\Delta L\right]$ $u_z = \left(1 - \dfrac{x}{\Delta L}\right)^2\left[\left(1 + 2\dfrac{x}{\Delta L}\right)u_{z1} + \dfrac{x}{\Delta L}\Psi_{y1}\Delta L\right]$ $\qquad + \left(\dfrac{x}{\Delta L}\right)^2\left[\left(3 - 2\dfrac{x}{\Delta L}\right)u_{z2}\right.$ $\qquad\qquad \left. - \left(1 - \dfrac{x}{\Delta L}\right)\Psi_{y2}\Delta L\right]$ (Shear Deformation Neglected)	Exact, no integration points
Mass [m]	$\dfrac{1}{2}\displaystyle\int_0^{\Delta L}\rho A\,\{u_x\ u_y\ u_z\ \Psi_x\ \Psi_y\ \Psi_z\}$ $\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&?\\0&0&0&\dfrac{\bar{I}_x}{A}&0&0\\0&0&0&0&\dfrac{\bar{I}_z}{A}&\rho\\0&0&0&0&0&\dfrac{\bar{I}_y}{A}\end{bmatrix}\begin{Bmatrix}u_x\\u_y\\u_z\\\Psi_x\\\Psi_y\\\Psi_z\end{Bmatrix}dx$ $\bar{I}_x = \bar{I}_y + \bar{I}_z$ $\rho, A, \bar{I}_x, \bar{I}_y, \bar{I}_z$ constant	$u_x = \left(1 - \dfrac{x}{\Delta L}\right)u_{x1} + \left(\dfrac{x}{\Delta L}\right)u_{x2}$ $u_y =$ same as for [k_G] $u_z =$ same as for [k_G] $\Psi_x = \left(1 - \dfrac{x}{\Delta L}\right)\Psi_{x1} + \left(\dfrac{x}{\Delta L}\right)\Psi_{x2}$ $\Psi_y = \left(1 - \dfrac{x}{\Delta L}\right)\left(1 - 3\dfrac{x}{\Delta L}\right)\Psi_{y1}$ $\qquad - \left(\dfrac{x}{\Delta L}\right)\left(2 - 3\dfrac{x}{\Delta L}\right)\Psi_{y2}$ $\qquad - \dfrac{x}{\Delta L}\left(1 - \dfrac{x}{\Delta L}\right)\left(\dfrac{u_{z2} - u_{z1}}{\Delta L}\right)$ $\Psi_z = \left(1 - \dfrac{x}{\Delta L}\right)\left(1 - 3\dfrac{x}{\Delta L}\right)\Psi_{z1}$ $\qquad - \left(\dfrac{x}{\Delta L}\right)\left(2 - 3\dfrac{x}{\Delta L}\right)\Psi_{z2}$ $\qquad - \dfrac{x}{\Delta L}\left(1 - \dfrac{x}{\Delta L}\right)\left(\dfrac{u_{y2} - u_{y1}}{\Delta L}\right)$ (Shear Deformation Neglected)	Exact, no integration points
Pressure Loading	$\displaystyle\int_0^{\Delta L}\left(p_y u_y + p_z u_z\right)dx$	u_y, u_z same as those for [k_G]	
Thermal Loading p'_T	$\displaystyle\int_0^{\Delta L}E\alpha\left[A\,\Delta T_0\dfrac{\partial u_x}{\partial x} - \bar{I}_z\dfrac{\Delta T_y}{h}\dfrac{\partial\Psi_z}{\partial x}\right.$ $\qquad\left. + \bar{I}_y\dfrac{\partial T_z}{b}\dfrac{\partial\Psi_y}{\partial x}\right]dx$ ΔT_y and ΔT_z = constant temperature differences in the y and z directions, respectively	u_x, Ψ_y, Ψ_z same as for [k_G] $\Delta T_0 = \Delta T_1\left(1 - \dfrac{x}{\Delta L}\right) + \Delta T_2\dfrac{x}{\Delta L}$	

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Stiffness and Mass Matrix Transformation

Matrix referred to Global Coordinates $= (T_1\ T_2\ T_3\ T_4)^T$ (Matrix referred to principal axes) $(T_1\ T_2\ T_3\ T_4)$

The matrix T_1 is the transformation matrix from principal to member axes:

$$T_{1(12 \times 12)} = \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}$$

$$R_{(3 \times 3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

with θ as defined previously. If the symmetric beam option is used, T_1 is set equal to the identity matrix I. Matrix T_2 is the transformation matrix for translation from element ends to offset nodes in member coordinates:

$$T_{2(12 \times 12)} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

$$S_{i(6 \times 6)} = \begin{bmatrix} 1 & 0 & 0 & 0 & -DZ_i & DY_i \\ 0 & 1 & 0 & DZ_i & 0 & -DX_i \\ 0 & 0 & 1 & -DY_i & DX_i & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the symmetric beam option, T_2 is taken as I.

Matrix T_3 is the transformation matrix for rotation from member axes to element axes.

$$T_3 = \begin{bmatrix} R_{x'x} & 0 & 0 & 0 \\ 0 & R_{x'x} & 0 & 0 \\ 0 & 0 & R_{x'x} & 0 \\ 0 & 0 & 0 & R_{x'x} \end{bmatrix}$$

Where the rotation matrix $R_{x'x}$ can be determined from Figure 17 as

$$R_{x'x} = \begin{bmatrix} \dfrac{L_A}{L_N} & \dfrac{\Delta_{y'}}{L_B} & -\dfrac{L_A \Delta_{z'}}{L_N L_B} \\ -\dfrac{\Delta_{y'}}{L_N} & \dfrac{L_A}{L_B} & \dfrac{\Delta_{y'} \Delta_{z'}}{L_N L_B} \\ \dfrac{\Delta_{z'}}{L_N} & 0 & \dfrac{L_B}{L_N} \end{bmatrix}$$
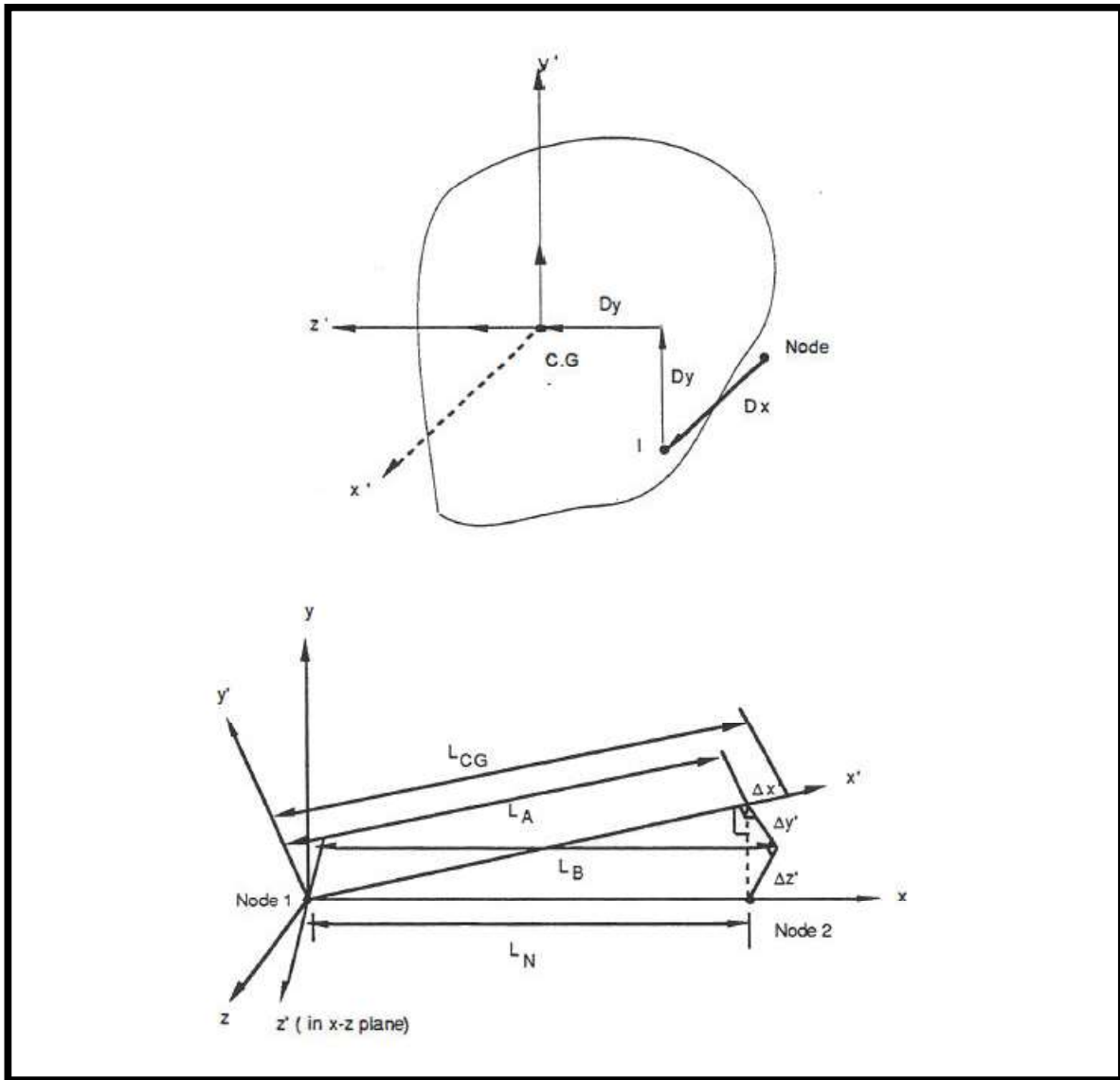
FIGURE 17

where

$$\Delta x' = DX2 - DX1$$

$$\Delta y' = DY2 - DY1$$

$$\Delta z' = DZ2 - DZ1$$

$$L_A = L_{CG} - \Delta x'$$

$$L_B = \sqrt{L_A^2 + y'^2} = \sqrt{L_N^2 - (\Delta z')^2}$$

For the symmetric beam option, T_3 is set equal to I.

The matrix T_4 is the transformation matrix from element to global coordinates.

$$T_{4(12x12)} = \begin{bmatrix} \overline{R} & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}$$

where

$$\overline{R}_{(3x3)} = \begin{bmatrix} n_x X & n_x Y & n_x Z \\ n_y X & n_y Y & n_y Z \\ n_z X & n_z Y & n_z Z \end{bmatrix}$$

When the third point is used

$$n_{xX} = \frac{X_2 - X_1}{\Delta L}, \quad n_{xX} = \frac{Y_2 - Y_1}{\Delta L}, \quad n_{xX} = \frac{Z_2 - Z_1}{\Delta L}$$

$$n_{zX} = \frac{N_{zX}}{N_z}, \quad n_{zY} = \frac{N_{zY}}{N_z}, \quad n_{zZ} = \frac{N_{zZ}}{N_z}$$

with

$$N_{zX} = (Z_3 - Z_1)(Y_2 - Y_1) - (Y_3 - Y_1)(Z_2 - Z_1)$$

$$N_{zY} = (X_3 - X_1)(Z_2 - Z_1) - (Z_3 - Z_1)(X_2 - X_1)$$

$$N_{zZ} = (Y_3 - Y_1)(X_2 - X_1) - (X_3 - X_1)(Y_2 - Y_1)$$

$$N_z = \sqrt{N_{zX}^2 + N_{zY}^2 + N_{zZ}^2}$$

$$n_{yX} = n_{zY}n_{xZ} - n_{zZ}n_{xY}$$

$$n_{yY} = n_{zZ}n_{xX} - n_{zX}n_{xZ}$$

$$n_{yZ} = n_{zX}n_{xY} - n_{zY}n_{xX}$$

Here X_i, Y_i, Z_i are the global coordinates of the three nodes. The matrix is used for both symmetric and unsymmetrical beams.

Force Matrix Transformation:

$$f = (T_1\ T_2\ T_3\ T_4)^T f'$$

Stress Calculation:

The stress is best determined in terms of forces at the member ends. These are given for principal axis coordinates as

$$
\begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{z1} \\ M_{x1} \\ M_{y1} \\ M_{z1} \\ P_{x2} \\ P_{y2} \\ P_{y2} \\ M_{x2} \\ M_{y2} \\ M_{z2} \end{Bmatrix} = k'_B (T_1\ T_2\ T_3\ T_4) \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{z1} \\ \Psi_{x1} \\ \Psi_{y1} \\ \Psi_{z1} \\ u_{x2} \\ u_{y2} \\ u_{z2} \\ \Psi_{x2} \\ \Psi_{y2} \\ \Psi_{z2} \end{Bmatrix} + E\alpha \begin{Bmatrix} A\Delta T_0 \\ 0 \\ 0 \\ 0 \\ \dfrac{\bar{I}_y}{b}\Delta T_z \\ -\dfrac{\bar{I}_z}{h}\Delta T_y \\ -A\Delta T_0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{\bar{I}_y}{b}\Delta T_z \\ \dfrac{\bar{I}_z}{h}\Delta T_y \end{Bmatrix}
$$

The axial stress is then determined as

$$\sigma_{x'i} = (-1)^i \frac{P_{x'i}}{A} - \frac{M_{z'i}y}{\bar{I}_z} + \frac{M_{y'i}Z}{\bar{I}_y}$$

To obtain the torsional stress, the twisting moments must be referred to the shear center for an unsymmetrical section. The end forces given above are then premultiplied by the transformation matrix.

$$T_5 = \begin{bmatrix} S_{SC} & 0 \\ 0 & S_{SC} \end{bmatrix}$$

with

$$S_{SC} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & DZS_c & DYS_c & 1 & 0 & 0 \\ -DZS_c & 0 & 0 & 0 & 1 & 0 \\ DYS_c & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e., the forces referred to the shear center are given by

$$P_{shear\ center} = T_5 P_{end}$$

For the symmetric beam option T₅ is set equal to I, but this is valid only for doubly symmetrical beams.

The torsional stress is then

$$\tau_{max} = \frac{(M_x)(CTOR)}{J}$$

Tapered Symmetrical Beams:

For a tapered doubly symmetrical beam, all matrices are calculated as for a doubly symmetrical uniform beam with the following equivalent properties:

$$A = \frac{1}{3}\left(A_1 + \sqrt{A_1 A_2} + A_2\right)$$

$$\bar{I}_y = \frac{1}{5}\left(\bar{I}_{y1} + \sqrt[4]{\bar{I}_{y1}^{-3}\bar{I}_{y2}} + \sqrt{\bar{I}_{y1}\bar{I}_{y2}} + \sqrt[4]{\bar{I}_{y1}\bar{I}_{y2}^{-3}} + \bar{I}_{y2}\right)$$

$$\bar{I}_z = \frac{1}{5}\left(\bar{I}_{z1} + \sqrt[4]{\bar{I}_{z1}^{-3}\bar{I}_{z2}} + \sqrt{\bar{I}_{z1}\bar{I}_{z2}} + \sqrt[4]{\bar{I}_{z1}\bar{I}_{z2}^{-3}} + \bar{I}_{z2}\right)$$

$$J = \frac{1}{5}\left(J_1 + \sqrt[4]{J_1^3 J_2} + \sqrt{J_1 J_2} + \sqrt[4]{J_1 J_2^3} + J_2\right)$$

For stress calculations, the actual stiffness at ends 1 and 2 are used.
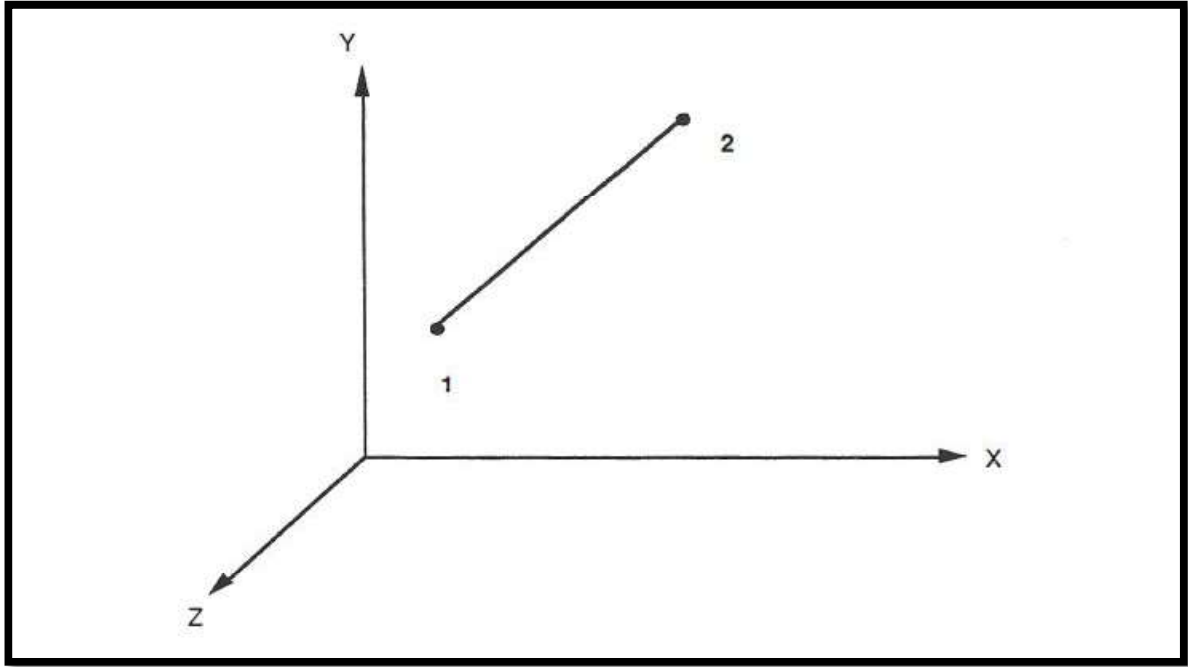
6.3. RBAR: 2-Node Rigid Bar[20]

FIGURE 18

Nodes 1 and 2 are connected by a rigid bar and are constrained to have rigid body relative displacements and the same rotation. The constraints in vector form are

$$\vec{\theta}_1 = \vec{\theta}_2$$

and

$$\vec{u}_2 - \vec{u}_1 = \frac{1}{2}(\vec{\theta}_1 + \vec{\theta}_2) \times (\vec{r}_2 - \vec{r}_1)$$

where $\vec{\theta}$ is the rotation vector, $\vec{u}$ the displacement vector, and $\vec{r}$ the radius vector. These constitute a set of six equations which are imposed via the penalty function method. The term

$$R = \frac{1}{2}\sum_{i=1}^{3}\left\{ E_i(\theta_{2i} - \theta_{1i})^2 + F_i\left[u_{21} - u_{1i} - \frac{1}{2}\sum_{j=1}^{3}\sum_{k=3}^{3}\varepsilon_{ijk}(\theta_{2j} + \theta_{1j})(x_{2k} - x_{1k})^2 \right] \right\}$$

is used as a variational function. The resulting stiffness matrix is included in the structural stiffness matrix. The quantities E_i and F_i are large numbers of the order of lE l0. The single subscripts i, j , k indicate the components of the various vectors and Eijk is the permutation symbol which is equal to zero if two or more of the subscripts are equal, and

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1, \qquad \varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$$
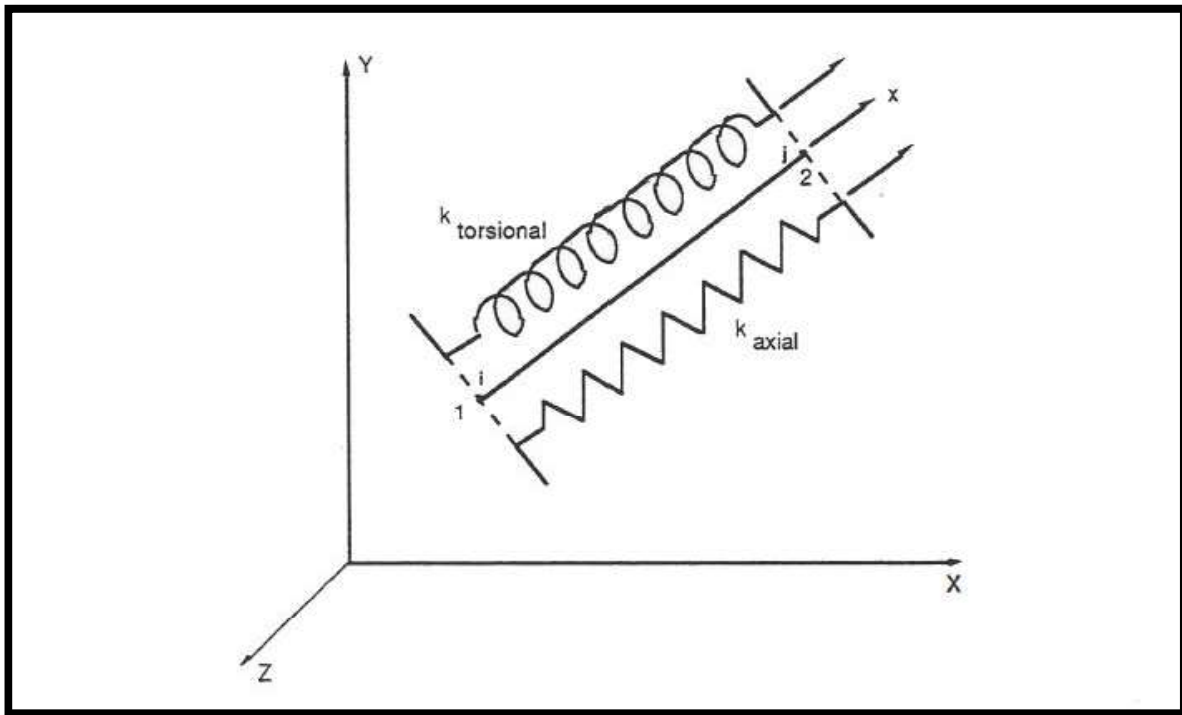
6.4.   SPRING: Spring Element

FIGURE 19

The element allows input of concentrated axial loads in the direction of the element axis and concentrated moments about the element axis which are proportional to the relative displacements of the nodes.

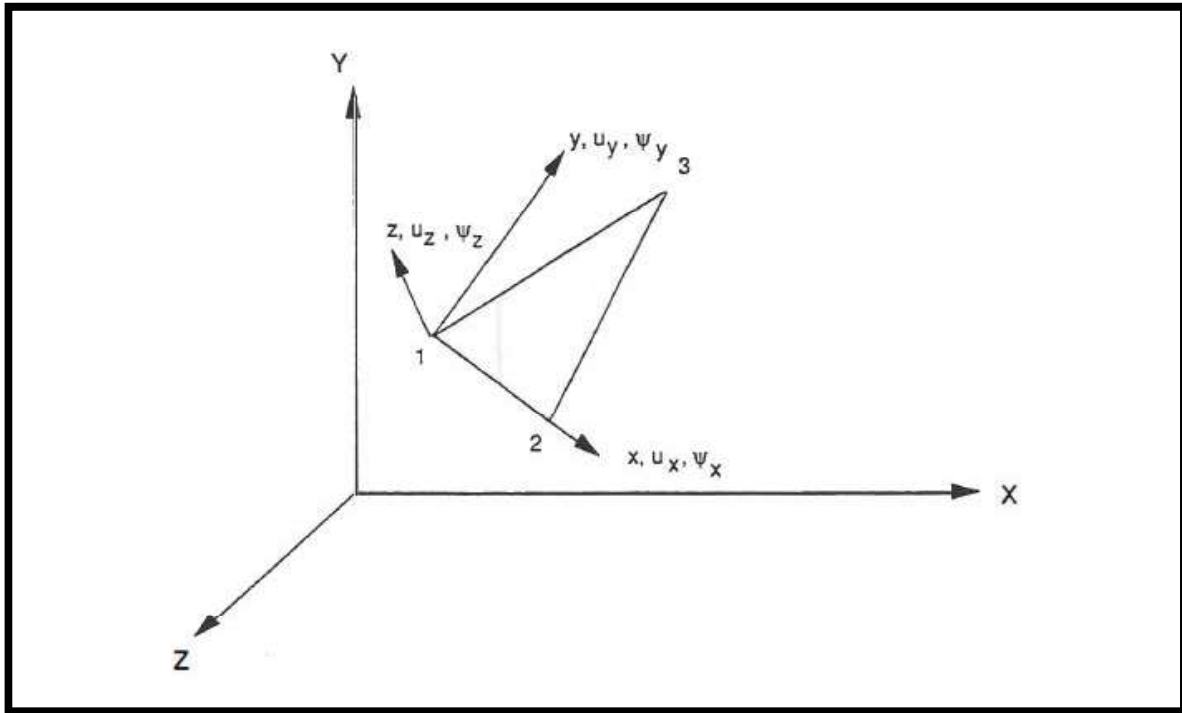| Matrix	Variational Function					Shape Functions	Integration
Linear Stiffness $[k'_B]$	$\frac{1}{2}\{u_{x1}\ \Psi_{x1}\ u_{x2}\ \Psi_{x2}\}\begin{bmatrix} k_{axial} & 0 & -k_{axial} & 0 \\ 0 & k_{torsional} & 0 & -k_{torsional} \\ -k_{axial} & 0 & k_{axial} & 0 \\ 0 & -k_{axial} & 0 & k_{torsional} \end{bmatrix}\begin{Bmatrix} u_{x1} \\ \Psi_{x1} \\ u_{x2} \\ \Psi_{x2} \end{Bmatrix}$					None	None

6.5. SHELL3T: Triangular Thick Shell[24,25,26]

FIGURE 20

Material is isotropic.

Matrix	Variational Function	Shop Functions	Integration
Linear Stiffness $[k'_{B_{18x18}}]$	Eqs. (2.19) to (2.21)	$u_x = u_{x1}L_1 + u_{x2}L_2 + u_{x3}L_3$ $\qquad + \frac{1}{2}y_3L_3[L_2(\Psi_{z3}$ $\qquad - \Psi_{z2})$ $\qquad - L_1(\Psi_{z1} - \Psi_{z3})]$ $u_y = u_{y1}L_1 + u_{y2}L_2 + u_{y3}L_3$ $\qquad + \frac{1}{2}[x_3L_1L_2(\Psi_{z1}$ $\qquad - \Psi_{z2})$ $\qquad + (x_2 - x_3)L_2L_3(\Psi_{z3}$ $\qquad - \Psi_{z2})$ $\qquad - x_2L_1L_2(\Psi_{z2} - \Psi_{z1})]$ $u_z = u_{z1}L_1 + u_{z2}L_2 + u_{z3}L_3$* $\Psi_x = \Psi_{x1}L_1 + \Psi_{x2}L_2 + \Psi_{x3}L_3$ $\Psi \;\; = \Psi_{y1}L_1 + \Psi_{y2}L_2 + \Psi_{y3}L_3$ * This is nominally the assumed displacement function. Derivation is	Exact for inplane energy; numerical with one integration point for bending energy

		modified by dividing displacements into bending and shear components and deleting bending portions of shear displacements. (L_1, L_2, L_3 area coordinates for triangle.)	
Geometric Stiffness [k'$_G$]	$\frac{1}{2} \iint\limits_{Area} \left\{\frac{\partial u_z}{\partial x} \ \frac{\partial u_z}{\partial y}\right\}$ $\begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{Bmatrix} \frac{\partial u_z}{\partial x} \\ \frac{\partial u_z}{\partial y} \end{Bmatrix} dx\,dy$	$u_z = u_{z1}L_1 + u_{z2}L_2 + u_{z3}L_3$	Numerical, 1 Gauss point
Mass [m']	$\frac{1}{2} \iint\limits_{Area} \rho \{u_x\ u_y\ u_z\ \Psi_x\ \Psi_y\}$ $\begin{bmatrix} \rho h & 0 & 0 & 0 & 0 \\ 0 & \rho h & 0 & 0 & 0 \\ 0 & 0 & \rho h & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12}\rho h^3 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12}\rho h^3 \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ u_z \\ \Psi_x \\ \Psi_y \end{Bmatrix} dx\,dy$	Same as for Linear Stiffness	Numerical, 1 Gauss point
Heat Transfer [k'$_T$]	$\frac{h}{2} \iint\limits_{Area} \left\{\frac{\partial T}{\partial x}\ \frac{\partial T}{\partial y}\right\}$ $\begin{bmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{bmatrix} \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{Bmatrix} dx\,dy$ $h \iint\limits_{Area} \rho c T \frac{\partial T}{\partial t}\, dx\,dy$	$T = T_1 L_1 + T_2 L_2 + T_3 L_3$	Numerical 1 point
[c'$_T$]	$h \iint\limits_{Area} QT\, dx\,dy$		3 points
{r$_Q$}			1 point
Pressure Loading	$\iint\limits_{Area} p\, u_z\, dx\,dy$	Same as for Linear Stiffness	1 point
Thermal Loading	$\iint E\alpha \left[\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\Delta T \right.$ $\left. + \frac{h^3}{12}\left(\frac{\partial \Psi_y}{\partial x} - \frac{\partial \Psi_y}{\partial y}\right)\Delta T_z\right] dx\,dy$	Same as for Linear Stiffness $\Delta T = \Delta T_1 L_1 + \Delta T_2 L_2 + \Delta T_3 L_3$ ΔT_z constant	1 point
Gravitational Loading	$\iint \left(g_x u_x + g_y u_y + g_z u_z\right) h\, dx\,dy$	Same as for Linear Stiffness	1 point

Transformation Matrix for Stiffness and Mass

$$k = T^T k' T$$

where

$$T_{(18 \times 18)} = \begin{bmatrix} \overline{R} & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{R} & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{R} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{R} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{R} \end{bmatrix}$$

$\overline{R}$ is same as rotation matrix $\overline{R}$ used in BEAM3D

Load Transformation

$$f = T^T f'$$

No transformation matrix for heat transfer
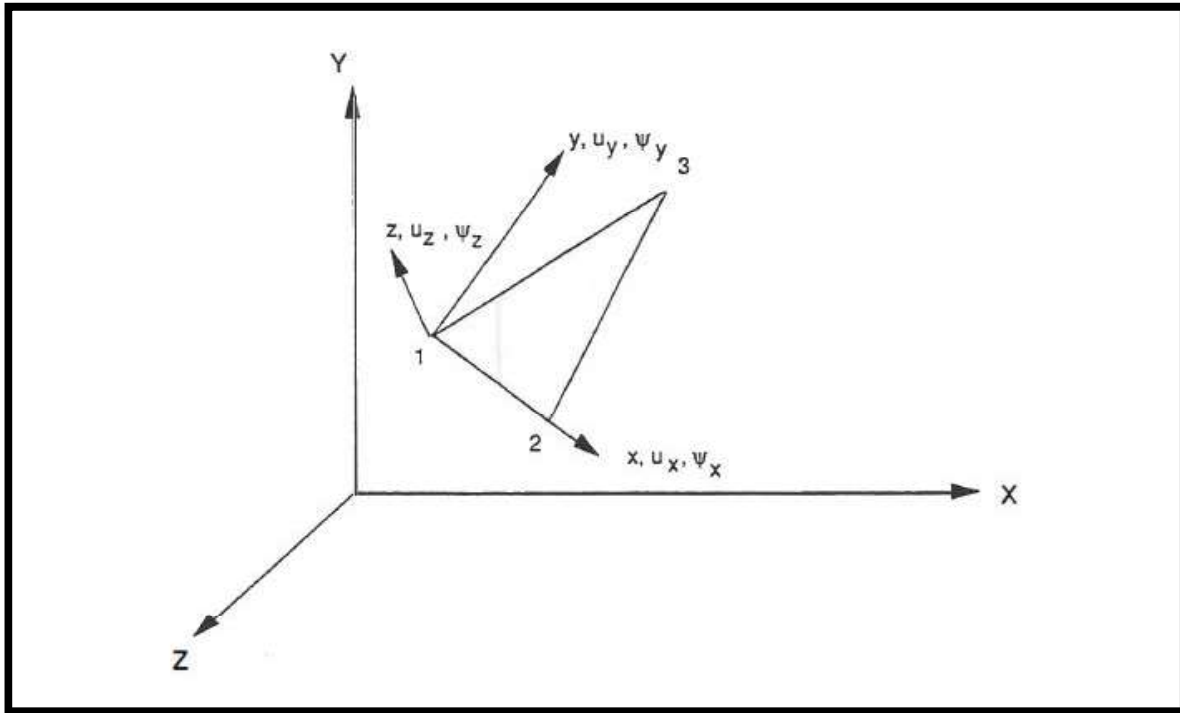
6.6.   SHELL3: Triangular Thin Shell

FIGURE 21

Material is isotropic.

Matrix	Variational Function	Shape Functions	Integration
Linear Stiffness	$\frac{1}{2}\iint\limits_{Area}\left\{\frac{\partial u_x}{\partial x}\ \frac{\partial u_y}{\partial y}\ \frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right\}$ $\begin{bmatrix} K & -vK & 0 \\ -vK & K & 0 \\ 0 & 0 & \frac{1-v}{2}K \end{bmatrix}\left\{\begin{array}{c}\frac{\partial u_y}{\partial y}\\ \frac{\partial u_x}{\partial x}\\ \frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\end{array}\right\}dx\,dy$ $+\frac{1}{2}\iint\limits_{Area}\left\{\frac{\partial \Psi_y}{\partial x}\ \frac{\partial \Psi_x}{\partial y}\ \frac{\partial \Psi_y}{\partial y}-\frac{\partial \Psi_x}{\partial x}\right\}$ $\begin{bmatrix} D & -vD & 0 \\ -vD & D & 0 \\ 0 & 0 & \frac{1-v}{2}D \end{bmatrix}\left\{\begin{array}{c}\frac{\partial \Psi_y}{\partial x}\\ \frac{\partial \Psi_x}{\partial y}\\ \frac{\partial \Psi_y}{\partial y}-\frac{\partial \Psi_x}{\partial x}\end{array}\right\}dx\,dy$	u_x, u_y same as for SHELL3T u_z is unspecified in interior along sides $u_z = \left(1-\frac{s}{l}\right)^2\left[\left(1+2\frac{s}{l}\right)u_{zi}+\frac{s}{l}\frac{\partial u_{zi}]}{\partial s}\right]$ $+\left(\frac{s}{l}\right)^2\left[\left(3-2\frac{s}{l}\right)u_{zj}\right.$ $\left.+1-\frac{s}{l}\frac{\partial u_{zj}]}{\partial s}\right]$ i = end 1, j = end 2, l = length of a side, s = distance along side from node I to node j	1 point integration

		$\begin{Bmatrix} \Psi_y \\ \Psi_z \end{Bmatrix} = L_1 2L_1 - 1 \begin{pmatrix} \Psi_{y1} \\ \Psi_{z1} \end{pmatrix} + L_2 2L_2$ $\qquad - 1 \begin{pmatrix} \Psi_{y2} \\ \Psi_{z2} \end{pmatrix} + L_3 2L_3$ $\qquad - 1 \begin{pmatrix} \Psi_{y3} \\ \Psi_{z3} \end{pmatrix}$ $\qquad + 4L_2 L_3 \begin{pmatrix} \Psi_{y4} \\ \Psi_{z4} \end{pmatrix}$ $\qquad + 4L_3 L_1 \begin{pmatrix} \Psi_{y5} \\ \Psi_{z5} \end{pmatrix}$ $\qquad + 4L_1 L_2 \begin{pmatrix} \Psi_{y6} \\ \Psi_{z6} \end{pmatrix}$ with Ψ_{yi}, Ψ_{zi} (i = 4, 5, 6) the rotations at the side midpoints. These are eliminated by imposing the Kirchoff conditions $$\frac{\partial u_z}{\partial x} + \Psi_y = 0$$ $$\frac{\partial u_z}{\partial y} + \Psi_x = 0$$ at the corners of the triangles, the conditions $$\Psi_s + \frac{\partial u_z}{\partial s} = 0$$ at the triangle midsides, with Ψ_s the rotation component in the direction of the side, and by imposing a linear variation of the normal rotation component along the sides, i.e., at the side midpoints $$\Psi_n = \frac{1}{2}\left(\Psi_{ni} + \Psi_{nj}\right)$$	
Geometric Stiffness	$\frac{1}{2} \iint\limits_{Area} \left\{ \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial u}{\partial y} \quad \frac{\partial u}{\partial x} \quad -\Psi_n \Psi_n \right\}$ $\begin{bmatrix} N_{xx}^0 & N_{xy}^0 & 0 & 0 & 0 & 0 \\ & N_{yy}^0 & 0 & 0 & 0 & 0 \\ & & N_{xx}^0 & N_{xy}^0 & 0 & 0 \\ & & & N_{yy}^0 & 0 & 0 \\ & & & & N_{xx}^0 & N_{xy}^0 \\ & & & & & N_{yy}^0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} \\ -\Psi_y \\ \Psi_x \end{Bmatrix} dx\,dy$	Same Shape Functions as for Linear Stiffness	Exact
	All other variational functions are identical with those for SHELL3T	Same Shape Functions as for Linear Stiffness	Exact, no integration points

Transformation Matrix

$$T = \begin{bmatrix} R & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{bmatrix}$$

R is identical with 3rd node transformation for BEAM3D.

Notation Table

A	area of a cross-section of a beam
$\overline{A}$	thermal expansion coefficient matrices (Eq. 1.37)
$\overline{\overline{A}}$	beam thermal expansion matrix
B_i	matrix of strain function of position within the ith finite element
C, C'	symmetric elastic coefficient matrices defining stress in terms of strain, referred to different coordinate systems (Eq. 1.36, 1.40a)
C_i	elastic constant matrix for i^{th} finite element
C_{tor}	coefficient for maximum shear stress of a section in torsion
$\overline{C}$	beam elastic stiffness matrix
$\tilde{C}$	flat plate elastic stiffness matrix
$\hat{C}$	elastic coefficient matrix for a conical shell
c	specific heat
$\overline{c}$	shifting parameter for frequency
D	isotropic plate bending stiffness
D_i	matrix of displacement functions of position within the ith finite element
D_s, D_ϕ	meridional and circumferential bending stiffnesses for an orthotropic conical shell
E, E'	square symmetric strain matrix of nine strain components at a point referred to different coordinate systems (Eqs 1.23, 1.26)
E	Young's modulus of isotropic material
ε_i	strain matrix for the ith finite element
E_{ii}	Young's modulus of orthotropic material with coincident material and coordinate axes relating direct strain in I direction due to direct stress in i direction
E_s, E_ϕ	Young's modulus in shell meridional and circumferential directions
$\overline{e}$	beam strain matrix (Eq. 2.12b)

ẽ   flat plate strain matrix (Eq. 2.20)

ê   strain matrix for a conical shell

F, F'   symmetric elastic coefficient matrices defining strain in terms of stress, referred to different coordinate systems (Eqs. 1.39, 1.40b)

F   assembled body force matrix

F$_i$ body force matrix for ith finite element

f$_i$ components of body force vector at a point acting in i direction

G   shear modulus of isotropic material $\left(G = \frac{E}{2(1+v)}\right)$

G$_{ij}$ shear modulus of orthotropic material with coincident material at coordinate axes giving shearing strain between lines in the I and j directions due to shearing stress component σ$_{ij}$

h   plate shell thickness; convective heat transfer or film coefficient

I   beam moment of inertia about an axis normal to plane of bending

I$_{yy}$, I$_{zz}$, I$_{yz}$ beam cross-sectional moments and product of inertia

I$_1$, I$_2$, I$_3$ stress invariants at a point (Eq. 1.20)

$\bar{I}_y$, $\bar{I}_z$ principal moments of inertia

J   beam cross-sectional torsion constant; Jacobian

J$_1$, J$_2$, J$_3$ strain invariants (Eq. 1.34)

K   assembled stiffness matrix

K   isotropic plate stretching stiffness $K = \frac{Eh}{1-v^2}$

K$_B$ structural stiffness matrix

K$_G$ geometric stiffness matrix

K$_S$, K$_\phi$ meridional and circumferential stretching stiffnesses for an orthotropic conical shell

k   thermal conductivity matrix

k$_i$ stiffness matrix for the ith finite element

k_x, k_y, k_z	thermal conductivities in Cartesian coordinate directions
$\bar{k}_y, \bar{k}_z$	shear strain energy correction coefficients for non-uniform distribution of shearing stress
M	assembled mass matrix
M_i	matrix relating nodal variables of ith element to the assembled matrix of all nodal variables
N	rotation matrix (square matrix of cosines of angles between coordinate axes in x and x' coordinate systems
n	outwardly directed normal to a plane passing through a point
n_i	outwardly directed normal to principal plane i at a point
P_k	transformation matrix used in Jacobi method (Eq. 3.19)
P_x, V_y, V_z	force components acting on a beam ends (Fig. 11)
$\bar{p}$	beam end force and moment matrix (Eq. 2.12a)
Q	internally generated heat flux
q	nodal variable matrix
q_i	matrix of nodal variables for i^{th} finite element
q_x, q_y, q_z	body heat flux in Cartesian coordinate directions
r, ϕ, z	coordinates of cylindrical coordinate system
S, S'	square symmetric stress matrices of nine components of stress vectors acting on different sets of three perpendicular planes about a point
$S_{\sigma i}$	square symmetric matrices of nine components of stress vectors acting on different sets of three perpendicular planes about a point
s, n	axes of conical orthotropy (Fig. 10)
s, ϕ, z	meridional, circumferential, and normal coordinates for a conical shell
T	temperature
T, M_y, M_z	moment components acting on beam ends (Fig. 11)
T_k	k^{th} transformation matrix for subspace iteration (Eq. 3.27a)

T_n	nodal temperature matrix for finite element
$\mathbf{T_\varepsilon}$	transformation matrix relating strain matrices ε and ε'
$\mathbf{T_\sigma}$	transformation matrix relating stress matrices σ and σ' (Eq. 1.14)
T_∞	fluid temperature
t	time
t_n	stress vector on a plane passing through a point and having the normal vector n
U	strain energy (Eq. 2.5)
u, v, w	flat plate reference plane displacements
u(s), w(s)	axisymmetric meridional and normal reference surface displacements for a conical shell
$\mathbf{u_i}$	displacement matrix for i^{th} finite element of deformable body
u_s, u_ϕ, u_z	conical shell displacements in direction of coordinate axes
u_x, u_y, u_z	deformation components in the direction of the Cartesian coordinate axes
u_{x0}	axial displacement of centroidal axes of an initially straight beam
$\bar{u}_x, \bar{u}_y, \bar{u}_z$	displacements of reference surface of plate or shell
V	volume of deformable body; kinetic energy (Eq. 3.3)
V_i	volume of the ith finite element
V_k	transformed eigenvector matrix for k^{th} round of subspace iteration (Eq. 3.27e)
W_i	numerical integration weighting constant for sampling point i in region
W	warping of beam cross-section
$\mathbf{x, x'}$	matrices of Cartesian coordinates of a point in space referred to different coordinate axes
x, y, z	coordinates of Cartesian coordinate system
α	thermal expansion coefficients for isotropic material
α, γ	terms in i^{th} and j^{th} rows and columns of P_k
α_{ij}	thermal expansion coefficients for direct (i = j) and shearing (i ≠ j) strains

Δ_0	displacement gradient matrix
ΔT	difference between actual temperature at a point and uniform temperature at which body is stress-free
δ_{ij}	Kronecker delta ($= 1$ if $i = j$, $= 0$ if $i \neq j$)
$\varepsilon, \varepsilon'$	column strain matrix of six independent components of E, E' (Eq. 1.27. 1.29)
ε_B	strain matrix for conical shells used in the determination of the geometric stiffness matrix
ε_i	principal strains
ε_{ij}	measure of change of length of a line originally in the i direction if i=j, a measure of the change of angle between two originally perpendicular lines in the i and j directions if i≠j
$\hat{\varepsilon}$	linear strain matrix for conical shells
θ	angle between coordinate systems having a common axis (Fig. 5); angle between radial axis and s axis of conical orthotropy in a body of revolution (Fig. 10)
λ	diagonal matrix of eigenvalues
λ	initial stress distribution proportionality factor
λ_k	diagonal eigenvalue matrix for k^{th} round of subspace iteration (Eq. 3.27d)
v	Poisson's ratio of isotropic material
v_{ij}	Poisson's ratio; ratio of strain in the i direction and strain in the j direction, due to stress in the j direction
$v_{s\phi}$	Poisson's ratio for orthotropic material
ξ, η	non-dimensional coordinates of a point in a regular element
Π	potential energy of elastic deformable body (Eq. 2.5)
Π_B	potential energy for buckling problems
Π_i	potential energy of ith finite element
ρ	mass of body per unit volume
σ, σ'	column stress matrix of six independent components of stress matrices S,S'

$\boldsymbol{\sigma_i}$	stress matrix i[th] finite element
σ_i	principal stresses at a point
σ_{ij}	stress component in j direction of stress vector acting on plane perpendicular to the I direction
σ_M	von Mises stress (Eq. 1.21a)
$\overline{\sigma}$	initial stress distribution matrix for buckling problems
τ_{oct}	shear stress on a plane making equal angles with respect to the principal axes
ϕ	matrix of eigenvectors arranged by columns
$\overline{\phi}_k$	eigenvector matrix for k[th] round of subspace iteration (Eq. 3.27e)
Ψ_x, Ψ_y, Ψ_z	small rotation components in the direction of the Cartesian coordinate axes
ω	vibration frequency
$\left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} =$	represents combination of two equations where each equation is defined by the corresponding terms on the same level after the bracket symbol

References

1. Timoshenko, S ., and Goodier, J. W.: Theory of Elasticity, 3rd Ed., McGraw-Hill Book Co., Inc., 1969.

2. Timoshenko, S., and Woinowsky-Krieger, S . : Theory of Plates and Shells, 2nd Ed., McGraw-Hill Book Co., Inc., 1959.

3. Timoshenko, S., and Gere, J. M.: Theory of Elastic Stability, 2nd Ed., McGraw-Hill Book Co., Inc., 196 1.

4. Runge, C.: Z . Math. u . Physik, vol. 50, p . 225, 1908.

5. Ritz, W.: übeer einer neue Methode zur Losung gewisser Variations probleme de mathematischer Physik, j .f.d. reine u. angew. Math. (Crelle), vol. 1 35, pp. 1 - 6 1 , 1909.

6. McCormac, J., and Elling, R.E.: Structural Analysis, Harper & Row, Publishers, 1 988.

7. Martin, H. C. , and Carey, G. F. : Introduction to Finite Element Analysis , Theory and

Application, McGraw-Hill Book Co., 1973.

8. Courant, R.: "Variational Methods for the Solution of Problems of Equilibrium and Vibrations", Bulletin of the Am. Math. Soc., vol. 49, pp. 1-23, 1943.

9. Lekhnitskii, S. G.: Theory of Elasticity of an Anisotropic Body, Mir Publishers, Moscow, 1981.

10. Washizu, K . : Variational Methods in Elasticity and Plasticity, 2nd Ed., Pergamon Press, 1975.

11. Cook, R. D.: Concepts and Applications of Finite Element Analysis, Second Edition, John Wiley & Sons, 198 1 .

12. Zienkiewicz, 0 . C . : The Finite Element Method, 3rd Ed., McGraw-Hill Book Co. (UK) Ltd., 1977.

13. Bathe, K. J., and Wilson, E. L.: Numerical Methods in Finite Element Analysis, Prentice-Hall, Inc., 197 6.

14. Weingarten, V. I., Ramanathan, R. K., and Chen, C. N.: Lanczos Eigenvalue Algorithm for Large Structures on a Minicomputer, Computers & Structures, vol. 1 6, no. 1-4, pp. 253-257, 1983.

15. Chowdhury, P. C.: The Truncated Lanczos Algorithm for Partial Solution of the Symmetric Eigenproblem, Computers & Structures, vol. 6, pp. 439-446, 1976.

16. Hughes, T. J. R.: The Finite Element Method, Prentice-Hall, Inc., 1987.

17. Seide, P.: Small Elastic Deformations of Thin Shells, Noordhoff Int. Publishing, 1975.

18. Przemieniecki, J. S.: Theory of Matrix Structural Analysis, McGraw-Hill Book Co., 1 968.

19. Hall, A. S., and Woodland, R. W.: Frame Analysis, John Wiley & Sons, 1961.

20. Lashkari, M., Liang, T.: Unpublished work, Structural Research.

21. Peano, A.: Hierarchies of Conforming Finite Elements for Plane Elasticity and Plate Bending, Comp. & Maths with Appls., vol. 2, pp. 2 1 1 -224, 1 976.

22. Szabo, B. A.: Some Recent Developments in Finite Element Analysis, Comp. & Maths with Appls., vol. 5, pp. 99- 1 15 , 1 979.

23. Babilska, I., Griebel, M., and Pitkaranta, J.: The Problem of Selecting the Shape Functions for a p-Type Finite Element, Int. J. Num. Methods in Eng., vol. 28, pp. 1891 - 1908, 1989.

24. Allman, D. J.: A Compatible Triangular Element Including Vertex Rotations for Plane Elasticity Analysis, Computers & Structures, vol. 19, no. 1-2, pp. 1-8, 1984.

25. Batoz, J. L., Bathe, K. J., and Ho, L. W.: A Study of Three-Node Triangular Plate Bending Elements, Int. J. Num. Methods in Eng., vol. 15, pp. 1771-1812, 1980.

26. Belytschko, T., Stolarski, H., and Carpenter, N.: A co Triangular Plate Element with one point Quadrature, Int. J. Num. Methods in Eng., vol. 20, pp. 787-802, 1984.

27. Cook, R. D. : On the Allman Triangle and a Related Quadrilateral Element, Computers & Structures, vol. 22, no. 6, pp. 1065-1067, 1086.

28. Bathe, K. J., and Dvorkin, E. N. : A Four-Node Plate Bending Element Based on Mindlin/Reissner Plate Theory and A Mixed Interpolation, Int. J. Num. Methods in Eng., vol. 21, pp. 367-383, 1985.

29. Bathe, K. J., and Ho, L. W. : A Simple and Effective Element for Analysis of General Shell Structures, Computers & Structures, vol. 13 , pp. 673-681, 1981.

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